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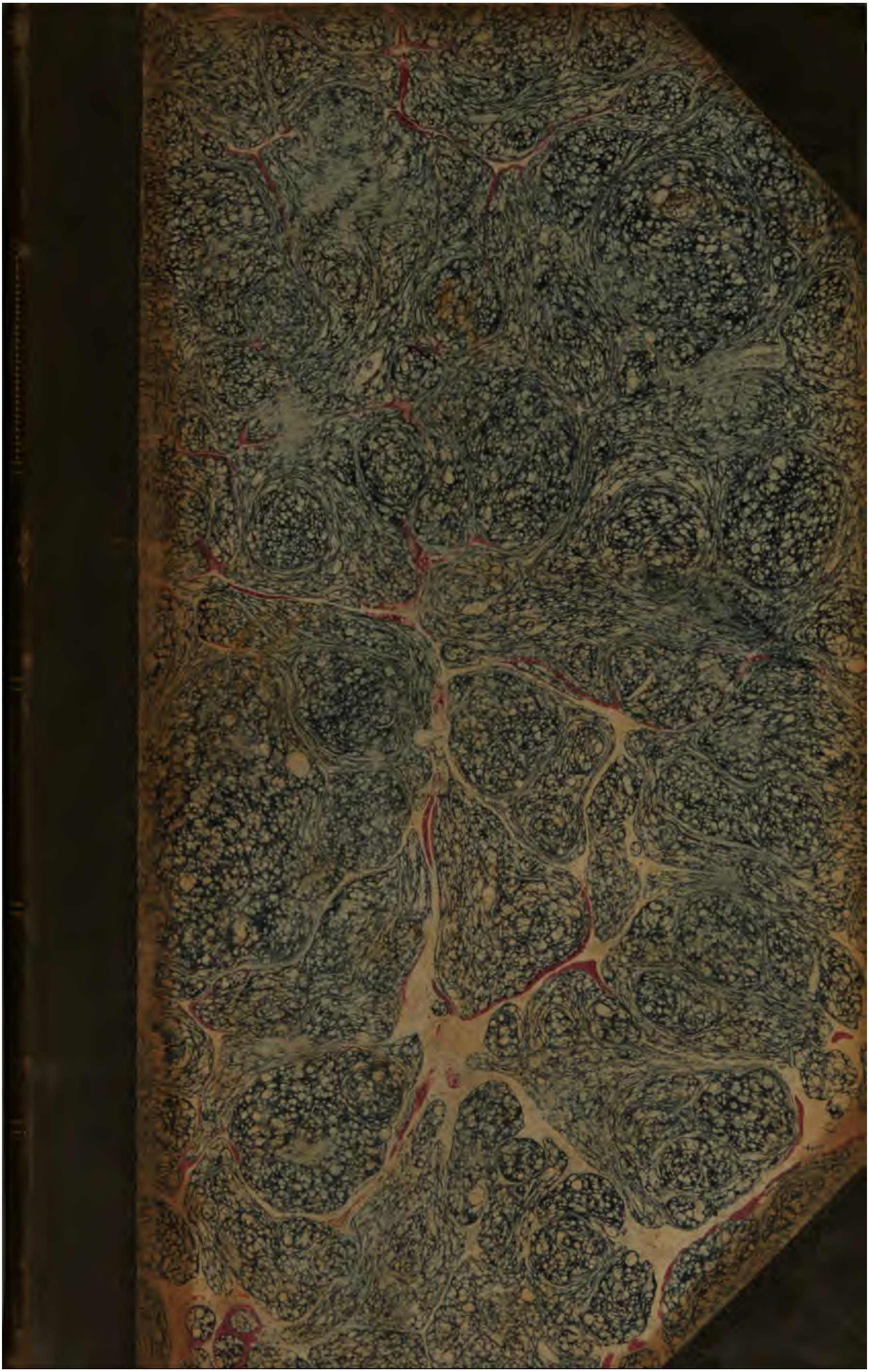

S. H. 1826

A
SHORT VIEW
OF
THE FIRST PRINCIPLES
OF THE
Differential Calculus.

By the Rev. ARTHUR BROWNE, M.A.

FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

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PREFACE.

OBJECTIONS have frequently been raised against the study of Mathematics, as pursued at the University of Cambridge. It has been urged, that since the truths, which this study unfolds, have no relation to the concerns of common life; and, that since a knowledge of the abstract properties of Geometrical figures, and Analytical symbols, has no tendency to make us either wiser or better than we were before, Mathematics must of all subjects be least calculated to prepare the minds of students, for whatever station in life they may hereafter be called upon to fill. It has been insinuated, that while, amid the variety of learned professions, it would have been impossible to select a subject, equally adapted to the wants of every student, we have, with singular impartiality, selected one, which is perfectly useless to them all. These objections have derived additional force, from its being observed, that the three years, which are spent at Cambridge, are, as far as education is concerned, the three best years of our life; and, that many, when they have completed their career at the University, become so occupied in the duties and business of their station, as to have no leisure for more useful and important studies. In answer to these objections it has been stated, that those young men, who annually resort to Cambridge, for the benefit of a University education, are mostly intended for one of the three learned professions, and that, in order successfully to prosecute the studies which belong to their profession, it is requisite, that their minds should be well habituated to thinking, acute in the detection of fallacious reasoning, capable of retaining and arranging their ideas, and of conveying the truths, which are perceived, into other minds, by clear and legitimate argument.

Now, if we consider the nature of Mathematical pursuits, we cannot fail to perceive, how admirably they are fitted to discipline the mind, and to give it those various qualifications, which are required in the prosecution of other studies. While, in such sciences as Chemistry and Geology, the information we derive, depends less upon the exercise of our intellectual powers, than upon the judicious selection and arrangement of experiments, the science of Mathematics consists of pure and abstract reasoning; starting from the knowledge of a few truths, respecting the laws of motion, and the properties of matter, sufficiently established by experiment and observation, it enables us, by a series of legitimate inferences, to arrive at the knowledge of other truths, so vast in their nature, that we might well have imagined them to lie beyond the reach of science.

Though Mathematics may have been sometimes debased by being employed in the discovery of unimportant truths, and in the solution of useless, though curious, problems*; yet, considering the insight they have given us into the laws of nature, and into the manner, in which the bodies, which compose our system, are mutually affected, we cannot but regard them as wonderful evidences of the power of the human understanding. That planetary influence should in one sense exist, that bodies, separated by a distance to us inconceivable, should yet act upon each other, and that the same power, which gives to all substances upon the earth, a tendency to fall downwards, should be that, which guides the planets in their orbits, are truths well calculated to excite our astonishment; but it is still more astonishing, that the ingenuity and research of man, should ever have been able to discover them. These truths have, however, been discovered, and it is one of the great objects of the Cambridge system of education, to bring the minds of students into an acquaintance with the whole train of reasoning, upon which they are established. The advantages of this system are manifest. Though the investigation of these truths does not, as it might seem, lie beyond the limits of our mental faculties, yet, it must still be remembered, it exerts those faculties to the very utmost, and is only to be effected by means of such continued and laborious thinking, as is inconceivable to those, who are occupied in the business of the world. It is on this account, that it becomes so well calculated to discipline the mind. If, in the studies of after life it is required, that the mind should be well inured to thought, powerful in the combination and arrangement of its ideas, and skilled in distinguishing between specious sophistry, and legitimate argument; who, I would ask, is so likely to rise to future eminence, as he, who in early life has made himself master of the works of Newton, and, by repeated and laborious thinking, familiarized himself with the whole train of reasoning and demonstration, by which the truth of his system is established? The Cambridge student, who is destined for one of the learned professions, may, perhaps, when he has finished his mathematical career, find that other competitors in the same profession, have applied themselves earlier to the requisite studies, and have, consequently, got the start of him with respect to time; but he will also find, that the habits of study and of thought, which he acquired at the University, will give him such a decided superiority over others of equal talent, who have not received the same mental discipline, that he will soon be able to rival and surpass them. Agreeably to this, I think it will be found, that the names of many of our best lawyers and soundest divines, rank high among the Cambridge Honors, and, (what is still more to the point), they are, I believe, seldom or never known to regret the time and labour which they once employed in mathematical pursuits.

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* These, I imagine, are the kind of problems, which Eubulus stigmatizes as being mere puzzles. To this class belongs the integration of many differentials and differential equations. Such questions require from the student the same kind of quickness and ingenuity, that is required in guessing rebusses and charades, and, like them are sometimes amusing from the neatness of their answers.

There is also another advantage arising from mathematical pursuits. In studying the writings of a man of genius, who, though his mind may not have been disciplined by severer studies, yet possesses in himself a copiousness of pleasing and instructive reflections, we are often, while the book is before our eyes, delighted with the ideas it suggests, but no sooner is it closed, than those ideas are forgotten, and we rise from the perusal of what once amused us, no wiser than we were before. Those writers, however, who have been habituated to mathematical thinking, and with minds thus strengthened, have entered upon the investigation of other subjects, are generally found to have the art of making their instructions more lastingly impressive. Their writings, (if the nature of the subject at all admits it,) are mostly characterized, not merely by the excellence and justness of each separate idea, but also by the obvious connection which exists between them, and by the skill, with which they are arranged. Hence it is, that, after we have risen from the perusal of the writings of such men, we can easily recal to our minds the ideas that were suggested; and, by frequently recalling them, familiarize ourselves with the whole train of thought, which it was the intention of those writings to convey. The utility of Mathematics, considered in this point of view, to those, who are intended for the law, is manifest; but they are, perhaps, still more useful to those, who are intended for the Church. The great fault of some modern discourses, is their want of arrangement; for, with all their artificial divisions, it will frequently be found, that there is no real connection between their several parts. Their authors, perhaps, have, by study and meditation, acquired an abundant store of religious ideas, and can express them in classical and elegant language; each separate sentence, therefore, in their discourses, is good, and, at the time of delivery, is felt by the hearers to be so; but, when the discourse is finished, there are few, who can remember the series of unconnected, though important truths, which they have just been hearing; and thus it is, that owing in a great measure to the want of a qualification, which the study of Mathematics is well calculated to give, the impression, which many a preacher produces, is neither so strong nor so lasting, as he could wish.

The utility, then, of mathematical pursuits, when considered as a discipline, by which the mind is fitted for other and more useful studies, is the principal ground upon which we can justify the great, and almost exclusive encouragement, which is given to them at Cambridge. That the student should be obliged to devote to them so large a portion of that time, which might be spent in pursuits, more closely connected with his future profession, is, we must confess, an evil; but, since the study of Mathematics has a tendency to strengthen and improve the thinking powers, we consider the advantage, as more than sufficient to counterbalance the evil.

If the justness of this view of the subject be admitted, there necessarily results the following conclusion: viz., that if any one branch of Mathematics exists, which is not calculated to produce these benefits, that branch ought immediately to be rejected as a subject of Academical Education. Now, I think it will be allowed by all those, who are acquainted with mathematical studies, that Analytics answer to this description, and that they, who devote their attention to this branch of Mathematics, will never find the advantages it confers, at all answerable to the time that it consumes.

I. Analytics have not, like Geometry, any tendency to inure the mind to those habits of thinking, which are so useful in the studies of after life. This, perhaps, is not strictly true of their first principles; for, when rigorously demonstrated, they are as well calculated to improve the mind in reasoning, as the proofs of Geometrical propositions; but, when these principles are once established, and certain rules and expressions are deduced from them, the application of these rules and expressions has no further influence in strengthening the powers of the understanding. Dr. Watson, the late Bishop of Landaff, has mentioned in his memoirs, that, during the period of his undergraduateship, he was in the habit, while walking, of going over in his mind the most intricate propositions in Euclid and Newton, and that, by frequent practice, he was able to accomplish this laborious task, without the assistance of the figures. It was, doubtless, this mental exercise, that laid the foundation of that strength of mind, and clearness of reasoning, by which he was afterwards so eminently distinguished.. Now, I would ask, does the study of Analytics afford the means of such discipline as this? Can the Analytical student retrace in his mind his former calculations, and, when pen, ink, and paper, are removed, deduce, by the mere power of his intellect those conclusions, which he has once found to be the result of his own working? Analytics have been well defined to be "the art of reasoning without thinking;" their superiority over Geometry consists in this, that, by using the symbols they afford, and working according to known rules and established principles, we may arrive at the knowledge of truths, to which Geometry alone would never have been able to conduct us. But this very circumstance, which constitutes their excellence in a mathematical point of view, renders them unfit for a subject of Academical education. We want to teach the students, who come to the University, to reason and to think, by calling their attention to a subject, which requires thought; but it is the peculiar use of Analytics, that they supersede the necessity of thinking.

II. Analytics have not, like Geometry, any tendency to make us better able to arrange and combine our ideas. They do, indeed, teach us to combine and arrange the Algebraical symbols we make use of, but they go no further. It is often the case, that when the Analytical student has been filling whole sheets with his calculations, and, in his own estimation, has been doing great things, his reasoning powers have all the while remained inactive; and though he may at length have arrived at some important conclusion, yet, during his progress, he has exerted no higher mental faculties, than those, which would have been called forth in working some of the long and tedious examples to the common rules of Arithmetic. This will, perhaps, account for the fact, that Analytics are so much the favourite branch of study with the rising generation of mathematicians. They know the important conclusions, to which Mathematics lead, and they wish to arrive at them themselves; but still the labour of thinking is disagreeably fatiguing, and they, therefore, prefer displaying the ingenuity with which they can manœuvre the Algebraic symbols, to pursuing a Geometrical demonstration through all its intricate details. But, while thus employed, it is manifest, that they are not in any way preparing themselves for the studies of after life; the difficulties of the law are not to be cleared, the doctrines of religion are not to be deduced by an ingenious application of symbols, but by frequent and careful meditation upon the ideas, which the study of those subjects suggests.

III. Analytics have not, like Geometry, any tendency to give us that power over our ideas, which enables us to express them in clear and perspicuous language. In this assertion I think I am fully borne out by the style, in which the works of some of the greatest Analytical writers are composed. The writings of Lacroix, for instance, are doubtless admirable in their way, and eminently useful to those, who intend to make Mathematics the study of their life. The estimation, in which his name is generally held, would lead us to imagine, that his works must exhibit a most favourable specimen of Analytical reasoning. But were we to appeal to the good sense of any one, who has diligently read the three large volumes, in which Lacroix has developed the principles of the Differential and Integral Calculus, and to ask him, whether he thought, that the attentive study of this work, would enable any one afterwards to reason more justly, and to express his ideas more clearly *upon other subjects*, I think he would immediately and unhesitatingly answer, No! It seems, in short, as if a familiar acquaintance with Analytical subtleties, had a natural tendency to destroy all relish for that plain and straight-forward method of reasoning, by which alone the truths of science can be made accessible to common understandings. And thus it is, that while English Mathematicians have been increasing the quantity of general knowledge by elementary treatises, concisely and clearly written, while, by the judicious manner in which they unfold the subject, they gradually prepare the mind of the student for the reception of deeper truths; the French, on the contrary, by the mysterious and profound nature of those demonstrations, which they apply even to the simplest theorems, seem, as if they laboured to shut the avenues of science against those, who are not prepared to devote their whole lives to its pursuit. The merit of the English Mathematicians consists in their being able to familiarize the remote; but it is the fault of the French, that they are too fond of estranging the familiar. Now, since it is in our nature to acquire those habits of thinking, and to imitate those modes of expressing our thoughts, which we find in the authors, with whose works we are most familiar, the study of Analytics must rather disqualify, than prepare us, for the duties of any future profession. He, who is called to the bar, must not only be able to understand the intricacies of the law, but he must also be able to adapt his explanation of them to the understandings of common men; he who is called to the Church, must set forth the doctrines of religion in such a manner, as to be understood by those, who know nothing of the refinements of learning. That student, therefore, who by studying the French Analytical writers, has at length acquired their obscure and confused manner of thinking, and their extreme fondness for those subtleties, which they deem elegant refinements, will, (if he ever enters the world) enter it with a mind that must be re-modelled, with a taste that must be corrected, and without a clearness of perception that must be acquired, before he can ever expect to become eminent, or even useful, in his profession. There is, however, something in the plain straight-forward method of Geometrical reasoning, peculiarly adapted to prepare the mind for the clear arrangement and expression of its thoughts. In composition, before we begin the construction of a sentence, we must first, as it were, take a distinct view of the ideas we are going to express, and it is the laborious thinking, which Geometry requires, that renders this species of mental exertion both habitual and easy.

The chief difficulty, which we meet with in the acquisition of knowledge, arises from our natural incapability of keeping our attention long fixed upon one subject, and of excluding those extraneous ideas, which are, from time to time, soliciting our notice. This difficulty, however, the study of Geometry is, in some degree, calculated to remove. He, who has long been in the habit of retracing, in his mind, the intricate propositions in Euclid and Newton, will, when he enters upon the investigation of other subjects, be better able, both to obtain for himself, and to communicate to others, a clear and distinct view of them, however complicated.

IV. Analytics have not, like Geometry, a tendency to enable us to form a comprehensive view of any subject, that may engage our attention. This will be more clearly seen, by stating the use and nature of these two branches of Mathematics. The noblest subject, to which this study can be applied, is the construction of the solar system, the description of the orbits traced out by the planets, and their satellites, and the investigation of the laws, by which the heavenly bodies attract, and are attracted, by each other. Upon this subject, the labours of Mathematicians have, for many years, been successfully employed; and it is manifest, that a general knowledge of their discoveries, must have a tendency to improve and to enlarge the mind. Of these discoveries, Geometry is calculated to give us the broad and general outline, and that too, by a series of demonstrations, where the justness and meaning of every step, may be distinctly seen; but, it is the province of Analytics, to carry us into the minuter details of mathematical truth; and, by the aid of Algebraic symbols, to enter into calculations, so involved, that the unassisted powers of the understanding, could never have worked them out. Geometry teaches us the nature of those orbits, in which the planets would have revolved, had they been affected only by the attraction of the Sun; but, since they attract each other, their mutual attractions though small in comparison with that of the Sun, still causes them, in some degree, to deviate from these orbits. Hence it is, that before science had reached its present perfection, the agreement, which subsisted between the Newtonian theory, as established upon Geometrical principles, and the observations, which Astronomers were then capable of making, was a sufficient confirmation of its truth; but, when by later discoveries in the construction of instruments, more accurate observations were made upon the motions of the planets, exacter values of their deviations were obtained, and it thenceforth became the object of Mathematicians to show, that these exacter values corresponded with theory. It was, therefore, necessary to discover the planetary deviations, by actual calculation: Geometry was insufficient for this purpose, and they had, therefore, recourse to Analytics. From this view of the subject, it manifestly follows, that though a knowledge of Analytics may be necessary for him, who intends to devote his whole life to Mathematics; yet, to the common student, they must be utterly useless. Though a familiar acquaintance with the general outlines of mathematical truth may enlarge the mind, yet, pursuing them into their minuter details, can have no tendency to give it any additional enlargement. Though the frequent contemplation of the construction of the solar system, as far as Geometry can explain it, may have such an influence upon the mind, as to enable it to form juster and clearer views upon other subjects, yet the exercise of calculating the various planetary deviations, which arise from disturbing forces, and of going through the long Algebraical operations which are necessary for that purpose, can have no such influence whatever.

Those, I fear, who are anxious to see the University of Cambridge, eminent in scientific pursuits, will hardly be brought to acquiesce in this view of the subject. They will ask, perhaps, shall we, while other Universities keep pace with the improvements in science, alone remain stationary? Shall we, while fresh discoveries are continually making, while mathematical truths are receiving new demonstrations, from a more general and refined Analysis, shall we alone adhere with a blind and instinctive attachment to the old methods, which, though used in the infancy of science, have long since been exploded? In answer to such questions, we have only to state the purpose, which Universities are intended to serve. This purpose, it must always be remembered, is not, that they may enlarge the bounds of scientific knowledge; but, that they may continually yield a supply of men, well qualified to fill the various offices, both in Church and State. Consequently, the comparative merits of different Universities, are not to be measured by the eminent talents, which some few of their distinguished members may display, but, by the average quantity of mental improvement, which all, who resort to them, derive from the studies they pursue. Hence it is, that, though among the Cambridge Honors, there are many to be found of the highest scientific eminence, yet, in boasting of the excellence of our system of education, we should refer with some feelings of triumph, to those who have risen in the law, and who, in their pleadings, manifest that capability of clearly arranging their ideas, which they acquired at Cambridge; and we should also refer, with feelings of still greater triumph, to those efficient clergymen, who are dispersed in great numbers through the country, and who, having received their education here, attribute much of their usefulness, as religious instructors, to the mental discipline, which they received from mathematical studies. If the justness of this view of the subject be admitted, it follows, that we are most effectually promoting the glory of our University, when we encourage those branches of Mathematics, which, though less splendid, are more useful; which, though less likely to enable the student to make further advances in scientific knowledge, are yet better calculated to prepare his mind for his future studies and pursuits.

If the opinions I have been stating on the comparative merits of the Geometrical and Analytical branches of Mathematics be correct, it necessarily follows, that Analytics are unfit for a subject of Academical education. They, who study them, must employ much time and labour in the pursuit, without receiving any adequate compensation, unless the pecuniary emoluments, to which their success may lead, is considered in that view. They must devote a considerable period of the three years they spend at Cambridge, to the integration of differentials, and the solution of differential equations, a species of mathematical trifling, in itself perfectly unprofitable, since it furnishes the mind with no useful ideas, does not bring into exercise the reasoning faculties, and, for the most part, leads to no important results. The mind, indeed, is naturally unwilling to leave the powerful machinery of Analysis, which enables it so easily to arrive at many of the most important mathematical results, and again to betake itself to the old laborious method of Geometrical demonstration; and it is on this account, that there seems great reason to fear, that unless some effectual barrier be raised against the introduction of French Mathematics, our University, which has long been, and still continues to be, the seat of sound learning and religious education, will, in time, become a mere school of useless subtleties, and Analytical refinements. Let it, however, be the boast of foreign Universities, that they excel in such refinements, that they enlarge the bounds of scientific knowledge, that they enter deeply into all the perplexing minutiæ of Physical and Mathematical investigations, and that, while thus employed, they occasionally light upon discoveries, which are curious in themselves, but useless to the majority of mankind;

> "Excudent alii spirantia mollius æra,
> Credo equidem; vivos ducent de marmore vultus
> Orabunt causas melius, *cælique meatus*
> *Describent radio*, et *surgentia sidera dicent*,
> Tu regere imperio populos, Romane, memento;
> Hæ tibi erunt artes, pacesque imponere morem
> Parcere subjectis, et debellare superbos."

But let us pursue a nobler course; let it be the peculiar care of this University, to train the students for those important duties, which they will be called upon to perform in after life. Let us always remember, that while we are engaged in our peaceful literary occupations, questions of great and momentous interest are continually agitating in the world, and that many, perhaps, of those, who are now pursuing their Academical studies, will, hereafter, from their rank in life, and from the situation they may fill, sustain an important part in the discussion. It is, therefore, of the utmost consequence, that, in early life, they should lay the foundation of those right habits of thinking, which will teach them what part to choose, and how to sustain it. In order that they may form correct views of the different subjects, to which their attention may, in after life, be directed, it is necessary, first, that they should have a full acquaintance with these subjects, in all their bearings and circumstances; and then, that they should have minds so disciplined in the habit of patient thinking, as to be in no danger of forming hasty conclusions; but, on the contrary, capable of deliberately estimating the comparative values of those various objections and arguments, which are sure to be brought forward in the discussion of any doubtful and important question. The first qualification can only be acquired in their intercourse with the world; the latter depends, in a great measure, upon the studies of their youth. Let it, therefore, be the care of this University, to direct the attention of its students to those subjects, which have a tendency to induce habits of correct reasoning; and since the Geometrical branch of Mathematics does, in a great degree, possess this tendency, since, by the strictness of its proofs, it is calculated to inspire the mind with a thorough contempt for the specious sophistry, so often met with in the world, it ought, surely, to form an essential part of Academical Education.

Again, while some, abstracted from the business of the world, have sufficient leisure to engage in those literary pursuits, which have no immediate bearing upon the wants and necessities of others; it should always be remembered, that the majority of mankind consists of those, who are entirely occupied in fulfilling the common duties of life, and who, in consequence, have neither time nor talents to acquaint themselves fully with that knowledge, upon which their present happiness, and eternal welfare, so mainly depend. It is this, which renders necessary the establishment of a public ministry; and the furnishing of the world with a constant supply of religious teachers, is one of the principal objects, for which Universities were furnished. Now, since it is manifest, that the diffusion of religious knowledge is of far greater importance, than the advancement of science, and since the usefulness and efficiency of those, who are hereafter to become public teachers, depend greatly upon the education, which they here receive: it should always be considered as the peculiar intention of Academical studies, to prepare the students for their future profession, and to give them the rudiments of that knowledge, which they are afterwards to communicate to others. In the explanation and enforcement of religious truth, that clearness of conception, and habitual justness of thought, which the study of our best Mathematical writers is well calculated to produce, cannot but be highly useful to every minister; both, because in his private studies, it will tend to secure him from those errors, into which men of warm imaginations and weak judgements are ever liable to fall; and also, because it will enable him to set forth the doctrines of religion in a manner so plain, as to be understood by the humblest of his hearers, and at the same time to establish them, by reasoning so just and convincing, as to satisfy the minds of those, who, though devoid of the refinements of learning, have yet shrewdness enough to detect a weak and fallacious argument.

While, however, mathematical studies are cultivated with this view, it should always be remembered, that we employ them only as means; that the knowledge they lead to, forms but a subordinate object; and that, consequently, in selecting from them the branches to be studied, we should be careful to choose,—not those, which enable us to arrive at the most important results with the least trouble, but those, which, by the rigorousness of their proofs, induce habits of correct reasoning, and from the nature of the difficulties they involve, are best calculated to exercise and to strengthen the powers of the understanding. In short, we should be more anxious to make the students good reasoners, than dexterous Analysts.

Having dwelt much longer than I intended upon the objections against the study of Analytics, it now remains, that I say something upon the intention of this little Work. It may seem strange, that an attack upon this branch of Mathematics, should constitute so large a part of the preface to a book, which is designed to treat upon that very subject; but the truth is, that whatever objections may lie against that study, when pursued too far, do not lie against its first principles; and the first principles are all that I have attempted to establish.

The first Section is upon the application of Algebra to Geometry, or rather, upon the representation of Geometrical figures by numbers. In this I have attempted to prove some truths, which have not generally been thought worthy of a demonstration. For my own part, I cannot help suspecting, that the sixth, seventh, and ninth propositions, have hitherto been considered as self-evident; not because they really are so, but because we have been so constantly in the habit of taking them for granted. In this and the following Sections, I have not made use of the consideration of velocity, as in fluxions, nor of limits, as in most treatises on the Differential Calculus, but have endeavoured to give a rigid demonstration of every proposition. The reason why Newton introduced his method of prime and ultimate ratios into Geometry, is, because most of his Geometrical demonstrations would have been insufferably tedious without it. But in Analytics, this objection does not exist; for when a few preliminary principles are laid down, the remaining truths are easily and shortly deduced from them. The great advantage, however, of discarding the consideration of limits is, that it renders every demonstration, (if not more rigid), at least more satisfactory. When the doctrine of Fluxions was first promulgated, there existed many individuals of no mean intellectual attainments, who were incapable of understanding it; and who, in consequence rejected, as illogical, the reasoning upon which it was founded. "The controversy, however, called forth Robins and Maclaurin, who showed, in the most satisfactory manner, that the grounds of Fluxions, according to the ideas of its great author, were defensible, and the investigations founded upon the strictest principles of reasoning*." Now it can hardly be expected, that the generality of students should be able fully to understand that train of Analytical reasoning, which, when first published, was too subtle for the comprehension of such men, as the ingenious author of the Analyst and his opponents. Agreeably to this, I think it will be found, that many of those, who display astonishing expertness in calculation, who can dispatch fluents and fluxional equations with inconceivable rapidity, have, at the same time, but very vague and confused ideas of the principles, by which they work. Some imagine, that quantities may exist, which are actually indefinitely small; others of superior metaphysical acuteness, consider the results they are conducted to, only as a species of approximation, not indeed perfectly accurate, but sufficiently so, for all the purposes of calculation. Since, then, the doctrine of limits is in its very nature so liable to be misunderstood, I cannot but hope, that the following Treatise will not be found entirely useless. Since my only object in composing it, was, to explain the reasoning, upon which the Differential Calculus is founded, I have only added, by way of illustration, a few of the easiest and most common-place examples.

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* See Vince's Scholium.

INTRODUCTION.

ON THE APPLICATION OF ALGEBRA AND ARITHMETIC TO GEOMETRY.

Though pure Geometry is sufficient to enable us to discover the abstract properties of magnitudes, yet in order to apply the knowledge thus obtained to practical purposes, we must have recourse to analytical calculation. Thus, for instance, if two sides and the included angle of a triangle are given, the remaining sides and angles may be assigned geometrically, but it would be impossible to calculate their actual value without the assistance of Trigonometry. Again, the orbits, which bodies describe when acted upon by external forces, may in some cases be ascertained by geometrical reasoning, but without the aid of the higher branches of Algebra, we could never have calculated the distances of the planets, their motions, or the forces, which act upon them, from the data, with which observation has supplied us. Hence arises the necessity of Algebra. In the measurement of plane rectilinear figures, Trigonometry is sufficient, but in investigating the lengths, areas, contents and surfaces of curvilinear figures, and in calculating the effects of irregular forces upon bodies, it is necessary to make use of the differential calculus. In Analytical calculations, all quantities, whether they are linear, plane, or solid figures, whether they are the forces, which act upon bodies, the velocities with which they move, or the times of their motion are represented by numbers.

The numerical representation of lines necessarily supposes, that there is some line, whose length is known, and expressed by unity. This line is called a linear unit. If a square, and a cube be described upon it, they are called respectively a plane and a solid unit.

Prop. I.

If any line be a multiple of a linear unit, it is capable of being represented by a whole number,

Suppose the given line to be divided into a number of parts, each equal to a linear unit, and let the number of these parts be (a). Then the length of the given line : the length of a linear unit :: (a) : 1, but the length of the linear unit is said to equal 1; therefore upon the same scale, the length of the given line may be said to equal (a), which is a whole number.

Prop. II.

If a given line be not a multiple of a linear unit, but has a common measure with it, the line may be represented by means of a fraction.

Since the given line, and the linear unit have both a common measure, let them both be divided by it, and let (a) be the number of times, which it is contained in the given line, and (b) the number of times, which it is contained in the linear unit. Then we have

$$\text{the given line} : \text{the common measure} :: a : 1,$$
$$\text{the common measure} : \text{linear unit} :: 1 : b,$$
$$\therefore \text{the given line} : \text{linear unit} :: a : b :: \frac{a}{b} : 1,$$

but linear unit $= 1$, $\therefore$ the given line $= \frac{a}{b}$, which is a fraction.

Prop. III.

If there be any line, which is not a multiple of unity, and has no common measure with it, then there is no number, whether whole or fractional, by which this line can be represented, but an approximation may be made to it by decimals, to any required degree of exactness.

The given line cannot be represented by a whole number, for if possible let it $= (a)$, then since the given line : linear unit :: $a : 1$, and that (a) is a multiple of unity, the given line must be a multiple of a linear unit, which is contrary to the supposition.

Neither can it be represented by a fraction, for if possible let it equal $\left(\frac{a}{b}\right)$; let it be divided into (a) equal parts, then

the whole line : one of these parts :: $a : 1$

$$:: \frac{a}{b} : \frac{1}{b}.$$

Hence each of these parts $= \frac{1}{b}$, and therefore the given line is a multiple of a line, whose length $= \frac{1}{b}$.

Again, let a linear unit be divided into (b) equal parts, then

the linear unit : one of these parts :: $b : 1$

$$:: 1 : \frac{1}{b};$$

$\therefore$ each of the parts, into which the linear unit is divided, $= \frac{1}{b}$, and therefore the linear unit is a multiple of a line, whose length $= \frac{1}{b}$, that is, the line represented by the number $\left(\frac{1}{b}\right)$ is a common measure of both the given line, and the linear unit; which is contrary to the supposition. Therefore the given line cannot be represented by any number, whether whole or fractional.

Still, however, an approximation may be made by decimals to any required degree of exactness. If we divide a linear unit into 10 equal parts, and find how often one of these parts is contained in the required line, the resulting number, with one place of decimals marked off, will represent a line, less indeed than the required line in length, but differing from it only by a quantity less than one tenth of a linear unit.

Again, if we divide a linear unit into 100 equal parts, and find how often one of these parts is contained in the given line, the resulting number with two places marked off as decimals, will represent a line, which differs from the required line by a quantity less than the $\frac{1}{100}$th part of a linear unit. And by dividing a linear unit into a 1000, or 10000 equal parts, we may find a number, which will represent it to a still greater degree of accuracy; and as there is no limit to the number of parts into which a linear unit can be divided, there can also be no limit to the degree of accuracy, with which incommensurable lines may be represented by numbers*.

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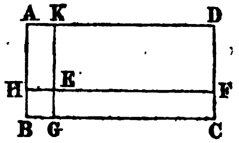
* That lines, admitting no common measure, may exist is manifest. Let the two sides of a right-angled isosceles triangle be each equal to a linear unit, then the square of the hypothenuse must

Prop. IV.

The number, which represents the area of a rectangle, is equal to the product of the numbers representing its adjacent sides.



Let $ABCD$ be a rectangle, and let the side $AB = a$, and $BC = b$, then the area of $ABCD = ab$.

Take BG, BH each $= 1$, draw HEF, GEK parallel to BC, and BA respectively, then $BHEG$ is a square, whose area = a plane unit = 1.

Also area of $BHEG$: area of rect. BF :: $BG : BC$
:: $1 : b$,
and area of rect. BF : area of rect. BD :: $CF : CD$
:: $BH : BA$
:: $1 : a$;
∴ area of $BHEG$: area of rect. BD :: $1 : ab$,
but area of $BHEG = 1$; ∴ area of rect. $BD = ab$.

Prop. V.

The number, which represents the content of a solid rectangular parallelopiped is equal to the product of the three numbers, which represent the lengths of its three sides.

Let $AEFB$ be a rectangular parallelopiped, and

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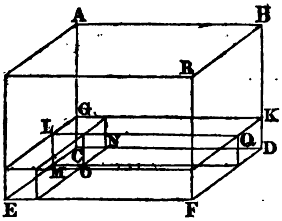
must equal 2, now as there is no number, whether whole or fractional, the square of which equals 2, the hypothenuse cannot be accurately represented in numbers. There are also no numbers, which will represent the circumference and area of a circle, whose radius equal 1, and indeed the lengths and areas of curves and the contents and surfaces of solids can seldom be accurately represented by numbers.



let the sides CA, CD, CE, which contain the solid angle at C, be equal to (a), (b), and (c) respectively, then the content of $AEF = abc$.

Take CG, CN, CM each equal to a linear unit, complete the cube GO, which will equal a solid unit, produce all its sides to the surface of the parallelopiped, and let them meet it in the points K, P, Q, M, &c. then

$$\begin{aligned}
\text{the cube } GO : \text{parallelopiped } CQ &:: \text{rect. } GN : \text{rect. } GD \\
&:: CN : CD \\
&:: 1 : b \\
\text{parallelopiped } CQ : \text{parallelopiped } KE &:: \text{rect. } GM : \text{rect. } GE \\
&:: CM : CE \\
&:: 1 : c \\
\text{parallelopiped } KE : \text{parallelopiped } AF &:: \text{rect. } GE : \text{rect. } AE \\
&:: GC : CA \\
&:: 1 : a
\end{aligned}$$

Therefore compounding these proportions, we have

$$\text{the cube } GO : \text{parallelopiped} : AF :: 1 : abc,$$

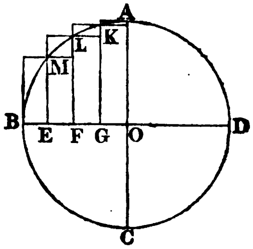
$$\text{but cube } GO = 1\text{; } \therefore \text{parallelopiped} = abc.$$

Cor. 1. If $a = b = c$, then we have the content of a cube whose side is $a = a^3$.

Prop. VI.

The numerical value of the content of a cylinder is equal to the product of the numbers, which respectively represent its length, and the area of one of its circular ends.

Let $ABCD$ be one of its circular ends; AC, BD



two diameters, intersecting each other at right angles at *O*. Divide *BO* into (n) equal parts *OG*, *GF*, &c.; erect the ordinates *GK*, *EL*, &c., and complete the rectangles *GA*, *OK*, *FK*, *GL*, &c.; and upon each of these rectangles let there be erected rectangular parallelepipeds, whose length equal the length of the cylinder. Let *A*, $4P$*, *a*, *b*, be the

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* This proposition might have been proved more shortly by the method of limits.

Making the same construction as before, it is manifest, that, when (n) is indefinitely increased, the content, to which the sum of the parallelepipeds, erected upon the rectangles *OK*, *GL*, *FM*, &c. approximates as its limit, is the quarter of the cylinder, standing upon the quadrant *BAO*. Hence, P is the limit of $S'a$.

Again. The area, to which the sum of the rectangles *OK*, *GL*, *FM*, &c. approximates as its limit, is the quadrantal area *BAO*; hence, A is the limit of S'; and, consequently, Aa is the limit of $S'a$.

Since, then, both P and Aa are equal to the quantity, to which $S'a$ approximates as its limits, when (n) is indefinitely increased, P must$=Aa$; and, therefore, $4P=4Aa$, as before.

The seventh proposition admits a similar demonstration. For it is manifest, that the limit of the sum of the rectangles bounding the content of the inner prism, is the curved surface of the cylinder, when (n) is indefinitely increased; hence, P is the limit of nca. Also, the circumference *ABCD* (which$=C$), is the limit of the perimeter of the polygon, which $= nc$; and, therefore, Ca is the limit of nca. Since, therefore, P and Ca are each equal to the limit, to which nca approximates, when (n) is indefinitely increased, $P = Ca$ as before.

The sixth proposition is also proved, (p. 131.), as a corollary to Prop. 44., and the seventh proposition admits a similar demonstration, as a corollary to Prop. 45.

From

numbers which respectively represent the quadrantal area BAO, the content of the cylinder, its length, and its radius BO. Let S represent the sum of the rectangles GA, FK, EL, &c.; then Sa will represent the sum of the parallelepipeds erected upon them; and will consequently be greater than P, which represents the quarter of the cylinder, standing upon the quadrant BAO. In the same way it may be shown, that, if S' represent the sum of the rectangles OK, GL, FM, &c., $S'a$ will be less than P. Hence, P lies between Sa and $S'a$. Consequently, if P and Aa be unequal, $P \sim Aa$ must be less than $Sa - S'a$. Now, $S - S' =$ sum of the rectangles AK, KL, &c. $=$ the rectangle $AG = \frac{b^2}{n}$ $\left(\text{for } AO = b, \text{ and } GO = \frac{b}{n}\right)$; hence, $Sa - S'a = \frac{ab^2}{n}$. Hence, $P \sim Aa$ is less than $\frac{ab^2}{n}$; and, consequently, (n) is less than $\frac{ab^2}{P \sim Aa}$, which is absurd, for (n) by the supposition may be assumed of any magnitude; hence, P and Aa, are not unequal; hence, $4P = 4Aa$, or the numerical value of the content of the cylinder

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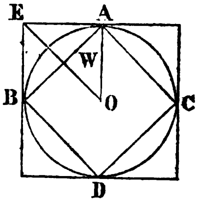
From the preceding propositions, we may easily find the areas of most rectilinear figures. Thus, if (a) be the base of a triangle, and (p) the perpendicular upon it from the opposite angle, the area of the triangle $= \frac{pa}{2}$. For if a rectangle be placed upon this base, having its altitude $= (p)$, and, consequently, lying between the same parallels with the triangle, its area is double that of the triangle, (by Prop. 41. Euclid.); consequently, twice the area of the triangle $= ap$; and, therefore, the area of the triangle $= \frac{ap}{2}$.

equals the product of the numerical values of its length and of the area of its circular end.

Cor. 1. The same demonstration would have applied, if *ABCD*, instead of being a circle, had been an area of any other form, whether rectilinear or curvilinear. Hence the numerical value of the content of a prism, having two of its opposite sides equal and parallel, and the remaining sides rectangles, is equal to the product of the numbers which represent its length and the area of one of its ends.

Prop. VII.

If (a), P, C, *numerically express the length, the curved surface, and the circumference of the circular end of a cylinder respectively, then we have the equation* P = Ca.



Let *ABCD* represent the circular end of the cylinder, in it describe an equilateral polygon *ABCD*, of (n) sides, where (n) is a number in the series 4, 8, 16; &c., and about it another equilateral polygon of the same number of sides, by drawing lines through the points *A, B, C, D*. Upon each of these polygons suppose a rectangular prism to be erected, whose length = (a), the length of the cylinder. Let c and c' numerically express the value of a side of the inscribed and circumscribed polygon respectively, then each of the rectangles bounding the content of the outer prism will $= c'a$, and their sum will $= nc'a$, and the sum of the rectangles bounding the content of the inner prism will $= nca$.

Now it is manifest, that the curved surface of the cylinder is less than the sum of the former rectangles, and greater than the sum of the latter, hence P lies between $nc'a$ and nca. Again, C manifestly lies between nc' and nc, and, therefore, Ca lies between $nc'a$, and nca, and since P and Ca lie both between $nc'a$ and nca, the difference between P and Ca, (if they are unequal) must be less than the difference between $nc'a$ and nca. Let, if possible, P be greater than Ca, then $P-Ca$ is less than $nc'a-nca$,

$$\begin{aligned}\text{but } nc'a : nca &:: c' : c\\ &:: EA : AW\\ &:: \text{rad} : \cos EAW\\ &:: 1 : \cos AOW \text{ or } \cos\frac{\pi}{n};\end{aligned}$$

$$\text{hence } nc'a = \frac{nca}{\cos\frac{\pi}{n}} = nca\,.\,\sec\frac{\pi}{n};$$

$$\therefore\ nc'a - nca = nca\,.\left(\sec\frac{\pi}{n} - 1\right);$$

$$\therefore\ P - Ca \text{ is less than } nca\,.\left(\sec\frac{\pi}{n} - 1\right),$$

but Ca is greater than nca, and $\tan\frac{\pi}{n}$ is greater than $\left(\sec\frac{\pi}{n} - 1\right)$; hence *a fortiori*,

$$P - Ca \text{ is less than } Ca\,.\tan\frac{\pi}{n};$$

$$\therefore\ \tan\frac{\pi}{n} \text{ is greater than } \frac{P-Ca}{Ca},$$

and consequently, if x be an arc, whose tangent $= \frac{P-Ca}{Ca}$,

$$\therefore \frac{\pi}{n} \text{ is greater than } z,$$

$$\text{or } n \text{ is less than } \frac{\pi}{z}.$$

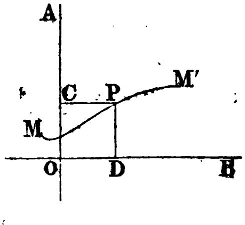
But it is manifest, that (n) may be assumed equal to some number in the series 4, 8, 16, &c., which is greater than $\frac{\pi}{z}$, which is absurd. Therefore P is not greater than Ca, and in the same manner it may be shown not to be less;

$$\therefore P = Ca.$$

Cor. 1. The same thing might have been proved in a similar manner, if the periphery of the end had been of an elliptic, or of any other form.

Prop. VIII.

If two straight lines be drawn at right angles to each other, and if the perpendicular distance of a given point from each of them, and the directions in which those distances are to be measured, be given, the position of the point may be found.



Let AO, OB be the given straight lines, take OC = the given perpendicular distance of the required point from OB, measuring it in the required direction, take OD = perpendicular distance of the point from AO, measuring it in the proper direction; complete the rectangle OP, then P is manifestly the required point.

Cor. 1. If in any curve, the relation between the two perpendicular distances of any point in it is such, that when one of these distances is known, the other may be found, and the direction in which it must be drawn ascertained; then the curve can be traced.

Def. 1. The lines *AO, OB* indefinitely produced are called axes.

2. The lines *OC, OD* are called co-ordinates to the point *D*.

3. The point *O*, where the axes meet, is called the first point.

4. If *MP* be any curve referred to the axes, *AO, OB* and *PD* be drawn perpendicular to *OB*, *PD* and *DO* are sometimes termed the ordinate and abscissa to the point *P* respectively.

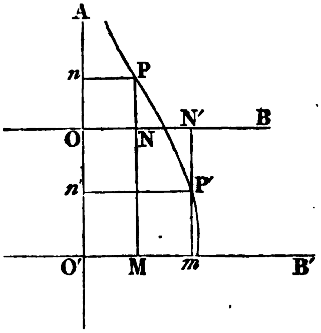
5. When the co-ordinate *OC* along the axis *OA* is measured upwards from *O*, it is said to be positive, and is therefore represented by a positive number; and the same is also true, when the co-ordinate *OD* along the axis *OB* is measured to the right of *O*.

6. An algebraic curve is one, where the relation between the two co-ordinates is represented by an equation. The ordinates and abscissas of algebraic curves are generally represented by the letters (y) and (x) respectively, and the equation between x and y is called the equation to the curve.

Prop. IX.

In any algebraic curve, if two quantities, (X) *and* (– Y) (*the latter being negative*), *being substituted* *in its equation for* (x) *and* (y) *respectively, will satisfy the conditions of the equation, the curve admits of an ordinate corresponding to* (X), *which is equal to* (Y), *but measured downwards from the first point.*

Let PQ be a curve, referred to the axes AO, OB; ON, On the co-ordinates to point P, and equal to (x) and (y) respectively. Let $ON' = X$, then if On' be measured downwards from O, and made $= Y$; On' is an ordinate to the curve, and completing the rectangle OP', P' is a point in the curve.



Take O', a point in AO beneath O, so that OO' is greater than Y; through O' draw $O'B'$ parallel to OB. Now the curve may be referred either to the axes AO, OB, or to the axes AO', $O'B'$. Take $O'n$, $O'M$, co-ordinates to the point P upon the latter supposition; then $O'M = ON = x$, let also $O'n = y'$, and $OO' = a$.

It is manifest, that while the curve lies above OB, the relation between y and y' is determined by the equation

$$y' = y + a, \quad \text{or } y = y' - a.$$

Now the nature of the curve, when referred to axes AO, OB is determined by an equation between (y) and (x), and if we substitute $(y' - a)$ for (y) in this equation, there will result another equation, which expresses the relation between (y') and (x), which equation will hold good, when the curve lies above

OB, but since the curve PP' is an algebraic curve, its nature is expressed throughout by the same equation, hence the equation between y' and x, will also hold good, when the curve lies below OB.

Now in the two equations between y and x, and between y' and x, make $x = ON' = X$. Then, if in the resulting equation between y and X, we substitute $(-Y)$ for (y), the terms of the equation are satisfied, or, in other words, $(-Y)$ is a root of the equation; but since the equation between y' and X is derived from that between y and X by substituting $(y'-a)$ for (y), the roots of this last equation must exceed the roots of the other by the quantity (a), therefore one root will be $(-Y+a)$, which root, since (a) is greater than Y, is positive, but less than (a), and because this quantity, when substituted for (y') in equation between y' and X, answers the conditions of it, it is therefore the co-ordinate with X, take therefore $O'm = X$, and $O'n' = a - Y$, and completing the rectangle $O'P'$, P' is a point in the curve, whose co-ordinates, when it is referred to the axes $O'A$, $O'B$ are $O'm$, On'. Again, since $O'n' = a - Y$ and $OO' = a$, we have $On' = Y$, that is, On' taken equal to Y, and measured downwards, is an ordinate corresponding to abscissa $ON' = X$.

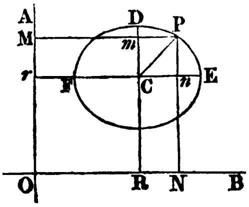
In the same manner it might have been shown, that, if Y and $(-X)$, (the latter being negative), when substituted for y and x respectively in the equation to the curve, fulfil its conditions, then a line measured along the horizontal axis to the left of the first point, and made $= X$, is an abscissa to the curve.

In a similar way it may also be shown, that if two negative quantities, $(-X)$ and $(-Y)$ when substituted for x and y in the equation to a curve, answer its conditions, then two straight lines, one $= X$, and measured along the horizontal axis to the left of the first point, and the other $= Y$, and measured downward from the first point along the perpendicular axis, will be co-ordinates to a point in the curve.

Cor. 1. Hence if the equation to any curve be such, that (y) can be generally found in terms of (x), the curve may be traced, or its form ascertained. For assuming (x) of all positive values, we may find all the corresponding values of (y), and measuring the positive values of (y) above the horizontal axis, and the negative below, the form of that part of the curve which lies to the right of the perpendicular axis is traced. In the same manner assuming (x) of all possible negative values, that form of the remaining part of the curve may be known.

Prop. X.

Having given the nature and properties of a curve, to find its equation.



Ex. 1. Let the curve be a circle, whose radius is (a), and the co-ordinates to whose centre are (α) and (β). Let *DEF* be the given circle. *AO*, *OB* the axes, take any point *P* in the circle, take also *OR*, *Or*, *ON*, *On* co-ordinates to the centre *C*, and the point *P* respectively;

$$\text{then } OR = \alpha,\ Or = \beta,\ ON = x,\ OM = y,$$

$$\text{and } Pn \text{ manifestly} = y - \beta, \text{ and } Cn = x - \alpha,$$

$$\text{but } Pn^2 + Cn^2 = CP^2,$$

$$\text{or } (y-\beta)^2 + (x-\alpha)^2 = a^2,$$

which is the general equation to the circle.

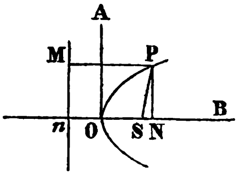
Cor. 1. If we suppose $\beta = 0$, and $\alpha = 0$, or the circle be referred to two diameters of its own as axes, we have

$$y^2 + x^2 = a^2, \quad \text{or } y = \pm\sqrt{a^2 - x^2}.$$

Cor. 2. If we suppose $\beta = 0$, but $\alpha = a$, or refer the circle to two axes, one of which is a diameter *FC*, and the other a straight line drawn from the point *F* perpendicular to it, we have

$$y^2 + (x - a)^2 = a^2, \quad \text{or } y = \pm\sqrt{2ax - x^2}.$$

Ex. 2. Let the curve be a parabola. Let *PO* be a parabola, whose focus is *S*, referred to two axes, one of which is its diameter, and the other a perpendicular drawn to it from its vertex.



Produce *BO* to *m*; make $mO = OS$, and through (m) draw *mM* parallel to *OA*; join *PS*, and from *P* draw *PM* perpendicular to *mM*, then by the nature of the parabola,

$$SP = PM,$$

now let $PN = y$, $NO = x$, $OS = a$;

then $SP^2 = SN^2 + PN^2 = (x - a)^2 + y^2$,

$$PM^2 = (x + a)^2;$$

whence $(x - a)^2 + y^2 = (x + a)^2$, or $y^2 = 4ax$.

Ex. 3. Let the curve be an hyperbola. Let the hyperbola be referred to its major and minor axes produced, as axes. Let the major axis $= 2a$, the minor $= 2b$, and distance between the foci $= 2c$, then calling z' and z the distances of any point from the outer and inner force respectively, we have

$$z'^2 = y^2 + (x+c)^2,$$

$$z^2 = y^2 + (x-c)^2,$$

$$z' - z = 2a,$$

$$b^2 = c^2 - a^2.$$

Whence eliminating c, z and z', we have

$$y^2 = \frac{b^2}{a^2} \cdot (x^2 - a^2), \quad \text{or } y = \pm \frac{b}{a} \sqrt{x^2 - a^2}.$$

SECT. I.

If two variable quantities so depend upon each other, that when one of them is given, the other must have some determinate value, the two quantities are said to be functions of each other.

Thus if (x) and (y) be two variable quantities, so dependent upon each other, that when (x) is assumed of any particular value, there are some corresponding values, which (y) must assume, in consequence of the relation subsisting between it and (x), then (y) and (x) are said to be functions of each other.

If, in any proposition or problem, one quantity is considered as a function of another, the latter quantity is sometimes by distinction termed the root of the former.

The relation between a function and its root, may sometimes be expressed by an equation, which gives the value of the function in terms of the root.

$$\text{Thus, let } y = \pm\sqrt{2ax - x^2},$$

$$\text{or} \quad y = (a+x)\,(b-x),$$

$$\text{or} \quad y = ax^{\frac{m}{n}} + bx^{\frac{r}{s}} - c.$$

In these equations, if we suppose the quantities (a), (b), (c), (m), (n), &c. to be constant, it is manifest, that if we substitute any quantity for (x), (y) must assume some particular values, in order that the equation may be satisfied. Hence, (y) is a function of (x) according to the definition; and since (y) can be expressed in terms of (x), it is called an explicit function of (x).

Sometimes the relation between a function and its root is expressed by such equations as these

$$y^3 \pm axy + x^3 = 0,$$

$$y^5 + (a+x)\,y^4 + (b+cx+kx^2)\,y^3 + gx^4y + x^5 = 0;$$

where, as before, to every value of (x), it is manifest, that there must be some corresponding values of (y), and (y) consequently is a function of (x); but, as (y) is not known in terms of (x), it is termed an implicit function of (x).

Sometimes the relation between a function and its root cannot be expressed by any equation, the number of whose terms are finite. Thus, if

$$y = \sin x,$$

$$\text{or}\quad y = \sin^{-1} x,$$

$$\text{or}\quad y = x + a \tan x,$$

$$\text{or}\quad y = \frac{a + bx - \sin x}{a - bx + \sin x},$$

$$\text{or}\quad y = \log x,$$

$$\text{or}\quad y = a^x;$$

it is still manifest, that every value of (x) has some value or values of (y) corresponding to it, and that consequently, (y) is still a function of (x); but, since, if the quantities $\sin x$, $\sin^{-1} x$, &c. are developed in terms of (x), the number of those terms will be infinite, there is no algebraical equation, which can represent the relation between (x) and (y), and they are therefore termed transcendental functions of each other.

Def. A series is said to converge, when every term in it is less in magnitude than the term before it.

Def. A series is said to diverge, when every term of it is greater in magnitude than the preceding term.

Thus the series

$$1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} - \&c.$$

is said to converge, because, though in the language of Algebra, $-\frac{1}{3}$ is greater than $-\frac{1}{2}$, yet in magnitude, the third term is less than the second, and the same is true of all the remaining terms.

Prop. XI.

If any quantity (X), *whose value depends upon another variable quantity* (x) *can be represented by such a series as*

$$A + Bx + Cx^2 + Dx^3 + \&c.$$

any term in this series may be made greater than the sum of all the terms that follow.

Let (K) be the value, which X assumes, when (k) is substituted for (x), then

$$K = A + Bk + Ck^2 + Dk^3 + \&c.$$

Now, since $X = A + x\ (B + Cx + Dx^2 + \&c.)$

it is manifest, that if for all the intermediate values of x between (k) and (o), the value of X remains finite, the series

$$B + Cx + Dx^2 + \&c.$$

must also remain finite, and consequently some constant quantity may be found, which will be greater than any value it assumes, while (x) diminishes from (k) to (o). Let G be this constant quantity,

then G is greater than $B + Cx + Dx^2 + \&c.$

or Gx than $x\,(B + Cx + Dx^2 + \&c.)$

But if (x) be assumed = to or less than $\frac{A}{G}$, Gx will be equal to, or less than A; and since A is either equal to or greater than Gx, and Gx greater than the sum of the terms, which follow (A), A must be greater than the sum of all the remaining terms of the series, that is, by making (x) less than either of the quantities (k) or $\frac{A}{G}$, the sum of all the terms in the series, except the first is rendered less than the first.

In the same manner it might have been shown, that by taking (x) sufficiently small, the series

$$Cx + Dx^2 + Fx^3 + \&c.$$

would have been rendered less than (B), and consequently

$$Cx^2 + Dx^3 + Fx^4 + \&c.$$

the remaining part of the series would then become less than the term Bx.

Cor. Hence it appears, that any series, representing the value of a finite quantity, of the form

$$Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$$

may, by diminishing (x), be rendered less than any assignable quantity.

In this proposition, we have introduced the limitation, that the series shall represent some variable function, which remains finite, while the quantity (x), upon which it depends, diminishes from a finite quantity to nothing, because it is not true of all series of the form

$$A + Bx + Cx^2 + Dx^3 + \&c.$$

that by diminishing (x), the value of any term may be rendered greater than the sum of all the terms that follow. Thus in the series

$$1 + x + 1 . 2^2 . x^2 + 1 . 2^2 . 3^3 x^3 + 1 . 2^2 . 3^3 . 4^4 . x^4 + \&c.$$

it is manifest, that however we may diminish (x), the coefficients increase so fast, that after a certain number of terms, the series must at length diverge. This series, however, if continued to infinity, cannot represent the value of any finite quantity. In the following part of this treatise, we shall, whenever we speak of series, only mean those, which represent some function of the variable quantity involved, and to which, therefore, the preceding proposition will apply.

Prop. XII.

If there be two series

$$A + Bx + Cx^2 + Dx^3 + \&c.$$
$$A' + B'x + C'x^2 + D'x^3 + \&c.$$

which are always equal to each other, whatever be the value of (x), *then the coefficients of the same power of* (x) *in each series are equal.*

If possible, let A be greater than A', then subtracting the under series from the upper, there results

$$(A - A') + (B - B')\, x + (C - C')\, x^2 + \&c.$$

which series must always be equal to nothing, whatever be the value of (x), now (x) may be made so small, that the first term $(A-A')$ becomes greater than the sum of all the rest, and in that case the series

$$(A-A')+(B-B')\,x+(C-C')\,x+\&c.$$

will be equal to some positive quantity, which is absurd; hence, A is not greater than A', and in the same manner it may be proved not to be less, therefore $A=A'$.

Again, since $A=A'$ the remaining part of the first series

$$Bx + Cx^2 + Dx^3 + \&c.$$

is equal to the remaining part of the second series,

$$B'x + C'x^2 + D'x^3.$$

And dividing by (x), the series

$$B + Cx + Dx^2 + \&c.$$

is equal to the series

$$B' + C'x + D'x^2 + \&c.$$

Hence, as before, $B = B'$.

In the same manner it might have been proved, that $C=C'$, and $D=D'$, and so on for the remaining coefficients of the series.

Prop. XIII.

If there be three series,

$$\mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \&c. \qquad (1).$$

$$\mathrm{A}' + \mathrm{B}'x + \mathrm{C}'x^2 + \mathrm{D}'x^3 + \&c. \qquad (2).$$

$$\mathrm{A}'' + \mathrm{B}''x + \mathrm{C}''x^2 + \mathrm{D}''x^3 + \&c. \qquad (3).$$

involving the positive integer powers of the same unknown quantity; and if the first be greater than the second, and the second greater than the third, then A′ *is either equal to one of the quantities* A *and* A″, *or else it lies between them.*

Since first series is greater than the second, we have, by subtraction,

$$A - A' + (B - B')\,x + (C - C')\,x^2 + \&c.$$

equal to some positive quantity. Now, if A' be greater than A, (x) may be assumed so small, that ($A' - A$) is greater than the sum of the remaining terms in the series. In that case, therefore, the series

$$A - A' + (B - B')\,x + (C - C'')\,x^2 + \&c.$$

must be negative, which is contrary to the supposition; hence, A' is not greater than A, and in the same manner it may be shown, that A'' is not greater than A'; therefore A' must either lie between A and A'', or else be equal to one of them.

Cor. 1. Hence, if $A = A''$, A' also equals A and A''. For A' is not greater than A, and not less than A''.

Cor. 2. The proposition would have held good, if we had supposed all the terms except the first, either in the first or third series, to have vanished. Thus, if a function of (x) represented by the series

$$A + Bx + Cx^2 + \&c.$$

lies between the constant quantity (A''), and the series

$$A' + B'x + C'x^2 + \&c.$$

then A lies between A' and A'', or else equals one of the quantities, and if $A' = A''$, A also equals A' and A''.

Cor. 3. Any series of the form

$$\pm A + Bx + Cx^2 + \&c.$$

may by diminishing (x), be made to have the same algebraic sign, that the first term has.

Cor. 4. If $A + Bx + Cx^2 + \&c.$ always $= 0$, whatever be the value of x, then

$$A = 0,\ B = 0,\ C = 0.$$

Def. If the root of a function increases by any given quantity, the function itself has a corresponding increase or decrease, which is called its increment or decrement, and represented by affixing the Greek letter Δ.

Thus if ϕx represents any function of x, and x increases by h, the corresponding increment of ϕx is represented by $\Delta . \phi (x)$;

$$\text{or } \phi(x + h) - \phi(x) = \Delta \phi(x).$$

The value of $\Delta\phi(x)$ depends both upon x and h, and if h be given, it depends upon x only, or is a function of x, and, therefore, if (x) be again supposed to increase by (h), $\Delta . \phi x$ may either increase or decrease, and its increment we call the second difference of the original function, and represent by $\Delta . \{\Delta\phi(x)\}$, or $\Delta^2 . \phi(x)$; and thus

$$\Delta . \phi(x+h) - \Delta\phi(x) = \Delta^2 . \phi(x).$$

In the same manner, we have

$$\Delta^2 . \phi(x+h) - \Delta^2 . \phi(x) = \Delta^3\phi x,$$

$$\text{and } \Delta^3 . \phi(x+h) - \Delta^3\phi(x) = \Delta^4\phi(x);$$

and generally,

$$\Delta^{n-1}.\phi(x+h)-\Delta^{n-1}.\phi(x)=\Delta^{n}\phi x,$$

where n is any whole number whatever.

PROP. XIV.

To prove that

$$\Delta^{n}\phi x=\phi(x+nh)-n.\phi\{x+(n-1)h\}$$
$$+\frac{n.(n-1)}{1.2}.\phi\{x+(n-2)h\}-\&c.$$

where the coefficients are the same as those in the expansion of $(1-x)^{n}$.

For $\Delta.\phi(x)=\phi(x+h)-\phi(x)$;

substituting $(x+h)$ for (x), we have

$$\Delta.\phi(x+h)=\phi(x+2h)-\phi(x+h);$$

hence, $\Delta.\phi(x+h)-\Delta\phi(x)$, or $\Delta^{2}\phi(x)$

$$=\phi(x+2h)-2\phi(x+h)+\phi(x).$$

From which we see that the formula holds good for $\Delta^{2}\phi(x)$. Now suppose the formula to hold good for $\Delta^{n-1}\phi(x)$; then

$$\Delta^{n-1}\phi(x)=\phi\{x+(n-1)h\}-(n-1)\phi\{x+(n-2)h\}$$
$$+\frac{(n-1)(n-2)}{1.2}.\phi\{x+(n-3)h\}-\&c.$$

and substituting $(x+h)$ for (x), we have

$$\Delta^{n-1}\phi(x+h)=\phi(x+nh)-(n-1)\phi\{x+(n-1)h\}$$
$$+\frac{(n-1)(n-2)}{1.2}.\phi\{x+(n-2)h\}-\&c.$$

And by subtraction, we find that

$$\Delta^{n-1}\phi(x+h)-\Delta^{n-1}\phi(x), \quad \text{or } \Delta^{n}\phi(x)$$
$$=\phi(x+nh)-n\,.\,\phi\{x+(n-1)h\}$$
$$+\frac{n\,.\,(n-1)}{1\,.\,2}\,.\,\phi\{x+(n-2)h\}-\&\text{c.}$$

From this we see, that if the formula holds good, as far as $\Delta^{n-1}\phi(x)$, it will also hold good in the next succeeding difference $\Delta^{n}\,.\,\phi(x)$, but it does hold good, as far as $\Delta^{2}\phi(x)$, it must, therefore, hold good as far as $\Delta^{3}\phi(x)$, and holding good for $\Delta^{3}\phi(x)$, it will also hold good for $\Delta^{4}\phi(x)$, and thus by successive inferences, we find it must be generally true.

Prop. XV.

To prove, that if n *be any integer,*

$$\phi(x+nh)=\phi x+n\,.\,\Delta\phi(x)+\frac{n.(n-1)}{1\,.\,2}.\Delta^{2}\phi(x)+\&c.$$

where the coefficients are the same as in the expansion $(1+x)^{n}$.

For $\phi(x+h)=\phi(x)+\Delta\phi(x)$ (*A*).

writing $(x+h)$ for (x), we have

$$\phi(x+2h)=\phi(x+h)+\Delta\phi(x+h) \quad (B).$$

subtracting equation (*A*) from equation (*B*), we have

$$\phi(x+2h)-\phi(x+h)$$
$$=\phi(x+h)-\phi(x)+\Delta\phi(x+h)-\Delta\phi(x)$$
$$=\Delta\phi(x)+\Delta^{2}\phi(x) \quad (C)$$

adding (*C*) to (*A*), we have

$$\phi(x+2h)=\phi(x)+2\Delta\,.\,\phi(x)+\Delta^{2}\phi(x).$$

Hence, this formula holds good as far as $\phi(x+2h)$. Now, suppose that it holds good as far as

$$\phi\{x+(n-1)h\}, \quad \text{then}$$

$$\phi\{x+(n-1)h\}=\phi(x)+(n-1)\Delta\phi(x)$$
$$+\frac{(n-1)(n-2)}{1.2}.\Delta^2\phi(x)+\&c. \qquad (D).$$

then writing $(x+h)$ for (x), we have

$$\phi(x+nh)=\phi(x+h)+(n-1)\Delta\phi(x+h)$$
$$+\frac{(n-1)(n-2)}{1.2}\Delta^2\phi(x+h)+\&c. \qquad (E).$$

Subtracting (D) from (E),

$$\phi(x+nh)-\phi\{x+(n-1)h\}$$
$$\left.\begin{aligned}&=\phi(x+h)-\phi x+(n-1).\{\Delta\phi(x+h)-\Delta.\phi x\}\\&+\frac{(n-1)(n-2)}{2}.\{\Delta^2\phi(x+h)-\Delta^2\phi(x)\}+\&c.\end{aligned}\right\}.$$

$$=\Delta\phi x+(n-1)\Delta^2\phi(x)$$
$$+\frac{(n-1)(n-2)}{1.2}.\Delta^3\phi(x)+\&c. \qquad (F).$$

Adding together (F) and (D),

$$\phi(x+nh)=\phi x+n\Delta\phi(x)+\frac{n.(n-1)}{1.2}.\Delta^2.\phi(x)+\&c.$$

From this we see, that if the formula holds good as far as $\phi\{x+(n-1)h\}$, it will also hold good as far as $\phi(x+nh)$. Suppose $n=3$, then we have seen that it holds good as far as $\phi\{x+(n-1)h\}$, or $\phi(x+2h)$; consequently it will hold good as far as

$$\phi(x+nh) \text{ or } \phi(x+3h).$$

In the same manner, making $n=4$, we may show, that it holds good as far as $\phi(x+4h)$, and thus, by successive inferences, we may prove it generally true.

PROP. XVI.

If $\phi(\mathrm{x})$ *be any function of* (x), *and if* (x) *increases by* (h), *then the corresponding increment of* $\phi(\mathrm{x})$ *may be expressed by a series, involving the integer powers of* (h), *connected with coefficients, which are functions of* (x).

For by the preceding proposition, if we suppose (x) to increase by (k), and (n) to be any integer, we have

$$\phi\,(x+nk)=\phi x+n\,.\,\Delta\phi\,(x)$$

$$+\frac{n.(n-1)}{1\,.\,2}.\Delta^2.\phi(x)+\frac{n.(n-1).(n-2)}{1\,.\,2\,.\,3}.\Delta^3\phi(x)+\&c.$$

where arranging the different terms according to the power of (n), we find

$$\phi\,(x+nk)=\phi\,(x)$$

$$+\,n\,.\left\{\Delta\phi\,(x)\;-\;\frac{\Delta^2.\phi\,(x)}{2}+\frac{\Delta^3.\phi\,(x)}{3}-\;\&c.\right\}$$

$$+\;n^2\left\{\frac{\Delta^2.\phi(x)}{1.2}-\frac{\Delta^3\phi\,(x)}{2}+\;\&c.\right\}$$

$$+\;n^3\left\{\frac{\Delta^3\phi x}{1.2.3}\;-\;\&c.\right\}$$

$$+\;\&c.$$

$$=\phi x+nS+n^2S'+n^3S''+\&c.$$

Where we assume S, S', S'', &c. to be equal to the coefficients of (n), (n^2), (n^3), &c. respectively; now let $nk=h$, then $n=\frac{h}{k}$, and

$$\phi(x+h)=\phi x+\frac{S}{k}.h+\frac{S'}{k^2}.h^2+\frac{S''}{k^3}.h^3+\&c.$$

Now, since $\Delta.\phi(x)$, $\Delta^2\phi(x)$, &c. are the successive increments of $\phi(x)$, taken upon the supposition of (x) increasing by (k), the values of S, S', S'', &c. are independent of (h). Again, the values of

$$\frac{S}{k}, \frac{S'}{k^2}, \frac{S''}{k^3}, \&c.$$

must be independent of the value of (n) or (k), or, in other words, they must remain the same, whatever submultiple (k) is assumed of (h), otherwise by changing the value of (n), we should find different series, which would represent the same quantity $\phi(x+h)$; consequently, the coefficients of the different powers of (h) being independent of both (h) and (k), are functions of (x) only; hence, assume them $=P$, Q, and R, &c. respectively, and we have

$$\phi(x+h)=\phi x+Ph+Qh^2+Rh^3+\&c.$$

Though this proposition is generally true, yet there may be particular cases, where it fails. Since P, Q, R, &c. are functions of (x), it may sometimes happen, that one of these quantities becomes infinite, when (x) assumes a particular value, and consequently, the expression for $\phi(x+h)$ will then become inapplicable.

Cor. 1. Hence, $\phi(x+h)-\phi(x)$,

$$\text{or}\quad \Delta.\phi(x)=Ph+Qh^2+Rh^3+\&c.$$

Def. The coefficient of the first power of the increment of (x) in the series expressing the value of $\Delta . \phi(x)$ is called the differential coefficient of $\phi(x)$, and may be represented by prefixing the letter f, thus, since

$$\Delta\phi(x) = Ph + Qh^2 + Rh^3 + \&c.$$

P is the differential coefficient of $\phi(x)$, and being itself a function of (x), is represented by $f.\phi(x)$.

Def. Since $f.\phi(x)$ is a function of (x), it is manifest, that, like other functions, it has a differential coefficient of its own; this differential coefficient is termed the second differential coefficient of the original function, and may be represented by $f.f.\phi(x)$, or $f^2\phi(x)$. In the same manner the differential coefficient of $f^2.\phi(x)$ is called the third differential coefficient of $\phi(x)$ and is represented by $f^3\phi(x)$. Thus every function of (x) may admit of a first, second, third, &c. .... nth differential coefficient, which coefficients are represented respectively by

$$f.\phi(x),\quad f^2\phi(x),\quad f^3\phi(x)\ldots.f^n\phi(x).$$

According to this notation, we have

$$\phi(x+h) = \phi x + f.\phi(x)h + Qh^2 + \&c.,$$

$$\text{or}\quad \Delta.\phi(x) = f.\phi(x)h + Qh^2 + \&c.$$

or representing $\phi(x)$ by (u),

$$\Delta(u) = fu.h + Qh^2 + \&c.$$

$$\text{Again,}\quad f.\phi(x+h) - f.\phi(x),$$

$$\text{or}\quad \Delta.f.\phi(x) = f^2\phi(x)h + Q'h^2 + \&c.,$$

$$\text{or}\quad \Delta.f(u) = f^2.(u).h + Q'h^2 + \&c.$$

$$\text{and } f^2.\phi(x+h)-f^2.\phi(x),$$

$$\text{or } \Delta.f^2.\phi(x)=f^3\phi(x)h+Q''h^2+\&c.,$$

$$\text{or } \Delta.f^2(u)=f^3.(u)h+Q''h^2+\&c.$$

And, in general,

$$f^{n-1}.\phi(x+h)-f^{n-1}.\phi(x),$$

$$\text{or } \Delta.f^{n-1}.\phi(x)=f^n.\phi(x)h+Q'''h^2+\&c.,$$

$$\text{or } \Delta.f^{n-1}(u)=f^n(u)h+Q'''h^2+\&c.$$

Def. The differential of a function is the first differential coefficient of that function, multiplied into some constant quantity, which we assume for the differential of its root.

The differential of a function is generally represented by prefixing the letter (d), thus, if $\phi(x)$ represents a function of (x), then the constant quantity, which we assume for the differential of (x), is represented by (dx), while $d.\phi(x)$ represents the differential of $\phi(x)$, and by the definition

$$d.\phi(x)=f.\phi(x).dx,$$

$$\text{and } u=\phi(x)\text{; then } du=f(u).dx.$$

Cor. Hence, $f.\phi(x)$, or $f(u)=\dfrac{d.\phi(x)}{dx}=\dfrac{du}{dx}$,

$$\text{and } \phi(x+h)=\phi(x)+\frac{d.\phi(x)}{dx}.h+Qh^2+\&c.$$

$$\text{and } \Delta.\phi(x),\text{ or } \Delta(u)=\frac{d.\phi(x)}{dx}.h+Qh^2+\&c.$$

$$\text{or } \frac{du}{dx}h+Qh^2+\&c.$$

PROP. XVII.

In the series, which expresses the value of the (n)*th difference of* $\phi(x)$, *the first term is the increment of* (x) *raised to the* (n)*th power, having for its coefficient the* n*th differential coefficient of* $\phi(x)$, *or*

$$\Delta^n \phi(x) = f^n \phi(x) h^n + \phi'(x) h^{n+1} + \&c.$$

Suppose this form to hold good as far as the $(n-1)$th difference, then

$$\begin{aligned} &\Delta^{n-1}\phi(x) \\ &= f^{n-1}\phi(x) h^{n-1} + \psi(x) h^n + \psi'(x) h^{n+1} + \&c. \end{aligned}$$

for x write $(x+h)$,

$$\begin{aligned} &\Delta^{n-1}.\phi(x+h) \\ &= f^{n-1}\phi(x+h) h^{n-1} + \psi(x+h) h^n + \psi'(x+h) h^{n+1} + \&c. \\ &= f^{n-1}\phi x . h^{n-1} + f^n \phi(x) h^n + G h^{n+1} + \&c. \\ &\qquad + \psi(x) h^n + f\psi(x) h^{n+1} + \&c. \\ &\qquad\qquad + \psi'(x) h^{n+1} + \&c. \\ &\qquad\qquad\qquad + \&c.; \end{aligned}$$

hence, $\Delta^{n-1}.\phi(x+h) - \Delta^{n-1}\phi(x)$,

or $\Delta^n \phi(x) = f^n \phi(x) h^n + G' h^{n+1} + \&c.$

Here we observe, that if this be true for any number $(n-1)$, it is also true for the next superior number (n); but it is true by the definition, when $(n-1) = 1$, it is therefore true for $n = 2$, and being true when $n - 1 = 2$, it is also true for $n = 3$, and thus, by successive inferences, we shall find it to be true for all superior numbers.

PROP. XVIII.

To prove generally, that

$$\phi(x+h)=\phi x+f\phi(x)h$$

$$+f^2\phi(x)\frac{h^2}{1.2}+f^3\phi(x)\frac{h^3}{1.2.3}+\&c.$$

We have seen, (Prop. 16.) that

$$\phi(x+h)=\phi x+Ph+Qh^2+Rh^3+\&c.$$

when $P=\frac{1}{k}\left\{\Delta\phi(x)-\frac{\Delta^2\phi(x)}{1.2}+\frac{\Delta^3\phi(x)}{1.2.3}+\&c.\right\}$,

$$Q=\frac{1}{k^2}\left\{\frac{\Delta^2\phi(x)}{1.2}-\frac{\Delta^3\phi(x)}{3}+\&c.\right\},$$

$$R=\frac{1}{k^3}\left\{\frac{\Delta^3\phi(x)}{1.2.3}-\&c.\right\};$$

where $\Delta\phi(x)$, $\Delta^2\phi(x)$, &c. represent the first, second, &c. differences respectively of $\phi(x)$, when (k) is the increment of (x), and some submultiple of h; then, expanding, we have

$$P=f.\phi(x)+G.k+\&c. \qquad \text{(Prop. 17.)}$$

$$-\frac{f^2\phi(x)}{1.2}.k+\&c.$$

Where, since the value of P remains the same, whatever submultiple (k) is of (h), all the terms in the series involving (k), must destroy each other; hence,

$$P=f\phi(x).$$

Again, by expanding the terms of the series, which $= Q$, we have

$$Q = \frac{f^2\phi(x)}{1.2} + G'k + \&c. \quad \text{(by Prop. 17.)}$$

$$- \frac{\Delta^3\phi(x)}{3} k + \&c.$$

where, as in the former case, all the terms, except the first, must be destroyed; hence,

$$Q = \frac{f^2\phi(x)}{1.2}.$$

In the same manner it may be proved, that

$$R = \frac{f^3\phi(x)}{1.2.3}.$$

Hence, substituting these values for P, Q, and R, we have

$$\phi(x+h) = \phi x + f\phi(x)h$$

$$+ f^2\phi(x)\frac{h^2}{1.2} + f^3\phi(x)\frac{h^3}{1.2.3} + \&c.$$

Where the law, by which the different terms of the series is ascertained, is manifest.

Cor. 1. If we represent $\phi(x)$ by (u), and suppose (u') to be the value, which (u) acquires, when (x) has increased by (h), we have

$$u' = u + f(u).h + f^2(u).\frac{h^2}{1.2} + f^3(u)\frac{h^3}{1.2.3} + \&c.$$

Cor. 2. $\phi(x) = \phi(o) + f.\phi(o).x$

$$+ f^2\phi(o)\frac{x^2}{1.2} + f^3\phi(o).\frac{x^3}{1.2.3} + \&c.$$

{where $\phi(o), f.\phi(o), f^2.\phi(o)$, &c. represent the values, which $\phi(x)$ and its differential coefficients assume, when $x=0$}, for

$$\phi(x+h)=\phi x+f(x).\phi h+f^2\phi(x)\frac{h^2}{1.2}+ \text{\&c.};$$

where making $x=0$, and writing (x) for (h), we have

$$\phi(x)=\phi(o)+f.\phi(o)x+f^2\phi(o)\frac{x^2}{1.2}+ \text{\&c.};$$

whence it appears, that, if the values which a function of x, and all its differential coefficients assume, when (x) is made $=0$, be either finite or $=0$, the function may be represented in a series involving the different integer positive powers of (x). This condition, however, is not fulfilled in such functions of (x) as

$$x^{\frac{m}{n}},\ \frac{1}{x^n},\ \log x,\ \sqrt{2ax-x^2},\ \text{cotan}\ x,$$

and these functions, therefore, cannot be represented by a series involving the positive integer powers of (x), while, on the other hand, such functions as

$$(1+x)^{\frac{m}{n}},\ \frac{1}{(a+x)^n},\ \log(1+x),\ \text{or cotan}\ (x+a),$$

may be represented by such a series.

Cor. 3. If, in the expansion of $\phi(x+h)$, there be any value of (x), which renders one of the differential coefficients infinitely great, at that point this theorem will fail. Thus, if we suppose

$$\phi(x)=x^2+\sqrt{x^2-a^2};$$

$$\text{then } f.\phi(x) = 2x + \frac{x}{\sqrt{x^2 - a^2}},$$

(by Prop. 30. Section 2.), where, if we make $x = a$, $f\phi(x)$ becomes infinite, and therefore the theorem fails; and if we substitute $(a + h)$ for (x), in the expression for $\phi\,(x)$, we find

$$\phi\,(a + h) = (a + h)^2 + \sqrt{(a + h)^2 - a^2},$$

$$= a^2 + 2ah + h^2 + h^{\frac{1}{2}}\sqrt{2a - h};$$

where, expanding $\sqrt{2a - h}$, and multiplying the series by $(h^{\frac{1}{2}})$, we find, that the series for $\phi\,(a + h)$ admits some fractional powers of (h).

Cor. 4. Hence, if a general method of finding the differential coefficients of functions were known, all functions might, by this series, be developed.

As this last proposition is of general use and application, its importance makes it worthy of another demonstration. Of all the various proofs, which have been employed to establish its truth, the following is one of the most common, and though long, is, perhaps, the easiest to understand.

Assume, what has been already proved, that

$$\phi(x + h) = \phi x + Ph + Qh^2 + Rh^3 + \&c.$$

$$\text{where } P = f\phi(x), \text{ and } Q, R, \&c.$$

are functions of x, and may therefore be represented by $\phi'(x)$, $\phi''(x)$, &c. respectively;

$$\text{then } \phi\,(x + h)$$

$$= \phi(x) + f\phi(x)\,h + \phi'(x)\,h^2 + \phi''(x)\,h^3 + \&c.$$

for h write $h+k$, then

$$\begin{aligned}
&\phi(x+h+k) \\
&= \phi(x) + f\phi(x)(h+k) + \phi'(x)(h+k)^2 + \phi''(x)(h+k)^3 \quad [+\&c. \\
&= \phi(x) + f\phi(x)h + \phi'(x)h^2 + \phi''(x)h^3 + \&c. \\
&\qquad + f\phi(x)k + 2\phi'(x)kh + 3\phi''(x)kh^2 + \&c. \\
&\qquad\qquad + \phi'(x)k^2 + 3\phi''xk^2h + \&c. \\
&\qquad\qquad\qquad + \&c.
\end{aligned}$$

Again, in the series for $\phi(x+h)$, substitute $(x+k)$ for (x); then

$$\begin{aligned}
&\phi(x+k+h) \\
&= \phi(x+k) + f\phi(x+k).h + \phi'(x+k)h^2 + \phi''.(x+k)h^3 \quad [+\&c. \\
&= \phi x + f\phi(x)k + \phi'(x)k^2 + \&c. \\
&\qquad + f\phi(x)h + f^2\phi(x)kh + S.k^2h + \&c. \\
&\qquad + \phi'(x)h^2 + f\phi'(x)kh^2 + S'k^2h^2 + \&c. \\
&\qquad + \phi''(x)h^3 + f\phi''(x)kh^3 + S''k^2h^3 + \&c. \\
&\qquad + \&c.
\end{aligned}$$

Now, in the former series, if we collect the terms, that involve only the first power of (k), we shall find them to be

$$f.\phi(x).k + 2\phi'(x)hk + 3\phi''(x)h^2k + \&c.;$$

hence, in the former series, the coefficient of (k) will be

$$f\phi(x) + 2\phi'(x)h + 3\phi''(x)h^2 + \&c. \qquad (A).$$

In the latter series, we shall, by collecting the terms that involve only the first power of (k), find the coefficient of (k) to be

$$f\phi(x) + f^2\phi(x)h + f\phi'(x)h^2 + \&c. \qquad (B).$$

Now, since $\phi(x+h+k)$ always $= \phi(x+k+h)$, the two series expressing them must always be equal, whatever be the value (k); hence, the coefficients of the corresponding powers of (k) in those series must be equal; therefore the series (A) must be equal to the series (B), and this too, whatever be the value of (h); hence, in the two series (A) and (B) the coefficients of the corresponding powers of (h) must be equal, and equating the coefficients, we have

$$2\phi'(x) = f^2\phi(x); \quad \therefore \ \phi'(x) = \frac{f^2\phi(x)}{1.2},$$

$$3\phi''(x) = f\phi'(x) = f.\frac{f^2.\phi(x)}{1.2} = \frac{f^3\phi(x)}{1.2};$$

$$\therefore \ \phi''(x) = \frac{f^3.\phi(x)}{1.2.3}.$$

And substituting these values, we have

$$\phi(x+h) = \phi(x) + f\phi(x)h$$
$$+ \frac{f^2\phi(x)}{1.2}h^2 + \frac{f^3.\phi(x)}{1.2.3}.h^3 + \&c.$$

Prop. XIX.

In the development of the series for $\phi(\mathrm{x}+\mathrm{h})$, *if any differential coefficient becomes constant, or independent of* (x), *all the succeeding differential coefficients vanish.*

Let $f^n\phi(x)$ be that differential coefficient, which first becomes constant, then since

$$f^n.\phi(x+h) = f^n.\phi(x)$$
$$+ f^{n+1}.\phi(x)\,h + f^{n+2}\phi(x)\frac{h^2}{1\,.\,2} + \&c.$$

$$\text{and } f^n.\phi(x+h) = f^n.\phi(x),$$

because the value of $f^n.\phi(x)$ is, by the supposition, independent of the value of (x), we have the series

$$f^{n+1}\phi(x)\,.\,h + f^{n+2}.\phi(x)\frac{h^2}{1\,.\,2} + \&c.$$

always $= 0$, whatever be the value of (h); hence,

$$f^{n+1}\phi(x) = 0, \quad \text{and } f^{n+2}\phi(x) = 0;$$

and so on for all the other differential coefficients.

Cor. 1. Hence, in this case,

$$\phi(x+h) = \phi(x) + f\phi(x)\,h$$
$$+ \frac{f^2\phi(x)}{1\,.\,2}\,.\,h^2 + \ldots\ldots\ldots \frac{f^n\phi(x)}{1\,.\,2\,.\,3\ldots n}\,.\,h^n,$$

and, therefore, the value of $\phi(x+h)$ can be expressed in a finite number of terms.

Prop. XX.

If two quantities be functions of each other, the differential coefficient of the first, considered as a function of the second, is equal to the reciprocal of the differential coefficient of the second, considered as a function of the first.

Let (x) and (y) be the two quantities; and let $y = \phi(x)$, and $x = \phi'(y)$,

$$\text{then } f.\phi(x) = \frac{1}{f.\phi'(y)}.$$

Let (h) and (k) be the corresponding increments of (x) and (y), then

$$y+k=\phi(x+h)$$

$$=\phi(x)+f\phi(x)\,h+f^2.\phi(x)\frac{h^2}{1.2}+\&c.;$$

$$\therefore\ k=f.\phi(x)\,h+f^2\phi(x).\frac{h^2}{1.2}+\&c.$$

In the same way it appears, that

$$h=f\phi'(y)\,k+f^2\phi'(y)\,\frac{k^2}{1.2}+\&c.$$

Hence, substituting this value of (h) in expression for (k), we have,

$$k=f\phi(x)\,.f\phi'(y)\,k+f.\phi(x)\,.f^2\phi'(y)\,.\,\frac{k^2}{1.2}+\&c.$$

$$+f^2.\phi(x).\{f.\phi'(y)\}^2\frac{k^2}{1.2}+\&c.;$$

$$\text{wherefore } f.\phi(x)\times f\phi'(y)=1;$$

$$\therefore\ f.\phi(x)=\frac{1}{f\phi'(y)}.$$

Cor. 1. Hence, any differential equation, which involves (x) and (y), and the first differential coefficient of (y) considered as a function of (x), may be transformed into a differential equation, involving (x) and (y), and the first differential coefficient of (x) considered as a function of (y), by substituting $\frac{1}{f.\phi'(y)}$ for $f.\phi(x)$.

Cor. 2. Hence, $f.\phi(x)\times f^2\phi'(y)$

$$+f^2.\phi(x)\times\{f\phi'(y)\}^2=0;$$

and, therefore,

$$f^2 \cdot \phi'(y) = \frac{-f^2 \cdot \phi(x) \times \{f\phi'(y)\}^2}{f \cdot \phi(x)}$$

$$= \frac{-f^2 \phi(x)}{\{f . \phi(x)\}^3};$$

$$\therefore\ f^2 \phi(x) = -f^2 \phi'(y) \times \{f \cdot \phi(x)\}^3 = \frac{-f^2 . \phi'(y)}{\{f \cdot \phi'(y)\}^3}.$$

Cor. 3. Hence any differential equation, which involves (x) and (y), and the first and second differential coefficients of (y) considered as a function of (x), may be transformed into an equation, involving (x) and (y), and the first and second differential coefficients of (x), considered as a function of (y), by substituting

$$\frac{1}{f\phi'(y)} \text{ for } f\phi(x), \text{ and } \frac{-f^2\phi'(y)}{\{f \cdot \phi'(y)\}^3} \text{ for } f^2 . \phi(x).$$

Prop. XXI.

If there be three quantities, functions of each other, the differential coefficient of the first, considered as a function of the third, is equal to the differential coefficient of the first, considered as a function of the second, multiplied by the differential coefficient of the second, considered as a function of the third.

Let (x), (y), (z) be three quantities, (h), (k), (l) their corresponding increments respectively; let

$$x = \phi(y) = \psi(z), \text{ and let } y = \psi'(z).$$

Then $h = f . \phi(y) . k + f^2 \phi(y) \dfrac{k^2}{1 . 2} + \&\text{c.}$ (A),

$$\text{and } h = f.\psi(z).l + f^2.\psi(z)\frac{l^2}{1.2} + \&c. \quad (B),$$

$$\text{and } k = f.\psi'(z)\,l + f^2.\psi'(z)\frac{l^2}{1.2} + \&c. \quad (C).$$

Whence, substituting the value of (k) as expressed in series (C) for (k) in series (A), we have

$$h = f.\phi(y) \times f\psi'(z)l + f.\phi(y) \times f^2.\psi'(z).\frac{l^2}{1.2} + \&c.$$

$$+ f^2\phi(y) \times (f.\psi'z)^2.\frac{l^2}{1.2} + \&c.$$

Hence, making the coefficient of (l) in this series equal to the coefficient of (l) in series (B), we have

$$f.\phi(y) \times f\psi'(z) = f\psi(z).$$

Cor. 1. Hence, $f\phi(y) = \dfrac{f.\psi(z)}{f.\psi'(z)}$.

Prop. XXII.

The differential coefficient of the sum of any number of functions is equal to the sum of their differential coefficients.

Let $\phi(x)$, $\phi'(x)$, $\phi''(x)$, be any functions of (x); and let $\psi(x)$ be equal to their sum, then

$$\psi(x) = \phi(x) + \phi'(x) + \phi''(x);$$

$$\therefore \psi(x+h) = \phi(x+h) + \phi'(x+h) + \phi''(x+h)$$

$$= \phi(x) + \phi'(x) + \phi''(x)$$

$$+ \{f\phi(x) + f\phi'(x) + f\phi''(x)\}\,h + \&c.$$

but $\psi(x+h) = \psi(x) + f\psi(x)\,h + \&c.$;

whence, equating coefficients, we have

$$f.\psi(x) = f.\phi(x) + f\phi'(x) + f\phi''(x).$$

The same proof would have applied, whatever had been the number of functions, $\phi(x)$, $\phi'(x)$, &c.

Cor. 1. Hence the differential of the sum of any number of functions is equal to the sum of their differentials; for since

$$f.\psi(x) = \frac{d.\psi(x)}{dx}, \quad f.\phi(x) = \frac{d\phi(x)}{dx},$$

$$f\phi'(x) = \frac{d\phi(x)}{dx}, \quad \text{and } f\phi''(x) = \frac{d\phi''(x)}{dx};$$

we have

$$\frac{d.\psi(x)}{dx} = \frac{d.\phi(x)}{dx} + \frac{d.\phi'(x)}{dx} + \frac{d\phi''(x)}{dx};$$

$$\therefore\ d.\psi(x) = d.\phi(x) + d.\phi'(x) + d.\phi''(x),$$

and if each of these functions had been represented by (s), (t), (v), respectively, we should have had

$$d(s+t+v) = ds + dt + dv,$$

where it is manifest, that the same will be true, whatever be the number of quantities, (s), (t), (v), &c.

Cor. 2. If any of the quantities, (s), (t), (v), &c. had been negative, the corresponding differential would have been negative also.

PROP. XXIII.

If two functions bear to each other an invariable ratio, all their corresponding differential coefficients are to each other in the same invariable ratio.

Let $\phi(x)$ and $\phi'(x)$ be two functions of (x), and let

$$\phi(x) : \phi'(x) :: 1 : a,$$

whatever be the value of (x);

$$\text{then } \phi'(x) = a \, . \, \phi(x),$$

$$\begin{aligned} \text{and } \phi'(x+h) &= a \, . \, \phi(x+h) \\ &= a\phi(x) + a \, . f . \phi(x) h \\ &\quad + a \, . f^2 . \phi(x) \frac{h^2}{1 \, . \, 2} + \&\text{c.} \end{aligned}$$

where, expanding $\phi'(x+h)$, and in the two series equating the coefficients of the corresponding powers of (h), we shall have

$$f . \phi'(x) = a \, . f . \phi(x), \quad f^2 \phi'(x) = af^2 \phi(x),$$

and so on for all the other differential coefficients.

Cor. 1. Hence, $d \, . \, \phi'(x) = ad \, . \, \phi(x)$,

$$\text{and } d \, . \, ax = adx, \quad \text{and } d \, . \frac{x}{a} = \frac{dx}{a};$$

$$\text{and } d \, . \, (a + bx) = bdx, \quad \text{and } d(a - bx) = -bdx.$$

Cor. 2. Hence, $\phi(x+h) = \phi x + \frac{d\phi x}{dx} . h$

$$+ \frac{d^2\phi(x)}{dx^2} . \frac{h^2}{1 \, . \, 2} + \frac{d^3\phi(x)}{dx^3} . \frac{h^3}{1 \, . \, 2 \, . \, 3} + \&\text{c.},$$

$$\text{for } \frac{d\phi x}{dx} = f\phi(x), \quad \text{and } \therefore d \, . f . \phi(x)$$

$$= d\left(\frac{d\phi x}{dx}\right) = \frac{d \, . \, d \, . \, \phi(x)}{dx} = \frac{d^2\phi(x)}{dx},$$

(dx) being considered as a constant quantity; and hence,

$$\frac{df\phi(x)}{dx} = \frac{d^2\phi(x)}{dx^2}; \text{ but } f^2\phi(x) = \frac{df.\phi x}{dx};$$

$$\text{and, therefore, } f'^2\phi(x) = \frac{d^2\phi(x)}{dx^2};$$

and the same may be proved of all the other differential coefficients.

Cor. 3. If $\phi(x)$ and $\phi(x+h)$ be represented respectively by (u) and (u'), then

$$u' = u + \frac{du}{dx}.h + \frac{d^2u}{dx^2}.\frac{h^2}{1.2} + \frac{d^3.u}{dx^3}.\frac{h^3}{1.2.3} + \&c.$$

Prop. XXIV.

If two functions have the same differential coefficient, their difference is constant.

Let $\phi(x)$ and $\phi'(x)$ two functions of (x) have the same differential coefficient $f\phi(x)$, then

$$\phi(x+h) = \phi(x) + f\phi(x)\,h + f^2\phi(x)\frac{h^2}{1.2} + \&c.$$

$$\text{and } \phi'(x+h) = \phi'(x) + f.\phi(x)h + f^2\phi(x)\frac{h^2}{1.2} + \&c.$$

therefore, by subtraction,

$$\phi(x+h) - \phi'(x+h) = \phi(x) - \phi'(x).$$

That is, the difference between the functions $\phi(x)$ and $\phi'(x)$ is not altered by increasing the value of (x) by (h), and, therefore, this difference remains constant.

Prop. XXV.

If the difference between two functions is constant, they have the same differential coefficient.

Let $\phi(x)$ and $\phi'(x)$ be the two functions of (x), and let

$\phi'(x) - \phi(x) =$ a constant quantity $= A$.

then $\phi'(x) = A + \phi(x)$;

$$\therefore\ \phi'(x+h) = A + \phi(x+h)$$

$$= A + \phi x + f\phi(x) . h + f^2\phi(x)\frac{h^2}{1.2} + \&c.$$

$$= \phi'(x) + f\phi(x)h + f^2\phi(x)\frac{h^2}{1.2} + \&c.$$

where $f.\phi(x)$ being the coefficient of (h) in the series for $\phi'(x+h)$ is also the differential coefficient of $\phi'(x)$, or $f.\phi(x) = f\phi'(x)$.

Cor. 1. Hence, all the other differential coefficients of these two functions are equal.

Cor. 2. Hence, in taking the differential coefficient of a function, the constant part, which is connected with the variable part only, by addition or subtraction, goes out.

Cor. 3. Since the differential of a function is equal to its differential coefficient multiplied into a constant quantity, which we assume for the differential of its root, it follows, that in differentiating a function, the constant part connected with the variable only, by subtraction or addition, goes out.

Cor. 4. Hence, $d.(x \pm a) = dx$,

and $d.\{\phi(x) + a\} = f.\phi(x)dx$.

PROP. XXVI.

If the sum of two functions be a constant quantity, their differential coefficients are the same in magnitude, but different in their algebraic sign.

Let $\phi(x)$ and $\phi'(x)$ be two functions of (x), and let $\phi(x)+\phi'(x)=A$, then substituting $(x+h)$ for (x), we have

$$\begin{aligned}
A &= \phi(x+h)+\phi'(x+h) \\
&= \phi x + f\phi(x)\,h + f^2\phi(x)\,.\,\frac{h^2}{1\,.\,2} + \&c. \\
&\quad + \phi' x + f\phi'(x)\,h + f^2.\phi'(x)\,\frac{h^2}{1\,.\,2} + \&c. \\
&= A + (f.\phi x + f.\phi' x)\,.\,h \\
&\quad + \{f^2.\phi(x) + f^2\phi' x\}\,.\,\frac{h^2}{1\,.\,2} + \&c.
\end{aligned}$$

Hence, the series

$$\{f\phi(x)+f\phi' x\}\,.\,h + \{f^2\phi(x)+f^2\phi'(x)\}\,\frac{h^2}{1\,.\,2} + \&c.$$

must always $=0$, whatever be the value of (h), and therefore, (Prop. 13. Cor. 4.).

$$f\phi(x) + f\phi'(x) = 0, \quad \text{or } f\phi(x) = -f\phi'(x),$$

$$\text{and } f^2\phi(x) + f^2\phi'(x) = 0, \quad \text{or } f^2\phi(x) = -f^2\phi'(x).$$

The same will be true of all the other coefficients.

COR. 1. Since $\phi'(x) = A - \phi(x)$, we have

$$d\,\{A - \phi(x)\} = -f\phi(x)\,.\,dx, \quad \text{and if } \phi(x) = x,$$

$$d\,(a - x) = -\,dx.$$

PROP. XXVII.

If there be two functions, the differential coefficient of their product is equal to the differential coefficient of the first function, multiplied into the second function, together with the differential coefficient of the second function, multiplied into the first function.

Let $\phi(x)$ and $\phi'(x)$ be two functions of (x), and let $\psi(x)$ be that function of (x), which is equal to their product, then,

$$\psi(x) = \phi(x) \times \phi'(x).$$

Writing $(x+h)$ for (x), we have

$$\begin{aligned} \psi(x+h) &= \phi(x+h) \times \phi'(x+h) \\ &= \{\phi(x) + f.\phi(x).h + \&c\}.\{\phi'x + f.\phi'(x).h + \&c.\} \\ &= \phi(x).\phi'(x) + \{\phi'(x).f\phi(x) + \phi(x).f\phi'(x)\}h + \&c. \end{aligned}$$

$$\text{but } \psi(x+h) = \psi(x) + f.\psi(x)\,h + f^2.\psi(x)\frac{h^2}{1.2} + \&c.$$

Equating the coefficients of the corresponding powers of (h), we have

$$f.\psi(x) = \phi'(x) \times f\phi(x) + \phi(x).f\phi'(x).$$

COR. 1. Hence, $\dfrac{d\psi(x)}{dx}$

$$= \frac{\phi'(x) \times d\phi(x)}{dx} + \frac{\phi(x) \times d\phi' x}{dx};$$

$$\text{and } \therefore d.\psi(x) = \phi'(x) \times d.\phi(x) + \phi(x) \times d.\phi'(x).$$

Cor. 2. If $\phi(x)$, and $\phi'(x)$ be represented by (s) and (t) respectively, we have

$$d.(st) = tds + sdt.$$

Cor. 3. From this we may find the differential of the product of any number of quantities, which are functions of the same quantity (x). Thus, in the above expression, write (tv) for (t), then

$$d.(stv) = tvds + s.d.(tv),$$

$$\text{but } d.(tv) = (\text{by Cor. 2.})\ vdt + tdv;$$

$$\therefore\ d.(stv) = tvds + svdt + stdv.$$

Again, for (v) write vw,

$$\text{then } d.(stvw) = tvwds + svwdt + std.(vw),$$

$$\text{but } d.(vw) = wdv + vdw;$$

$$\therefore\ d.(stvw) = tvwds + svwdt + stwdv + stvdw.$$

From whence it appears, that the differential of the product of any number of functions is equal to the sum of the products, which arise from multiplying the differential of each function into the product of all the other functions.

Def. A function is said to be at its maximum value, when its value is diminished, either by increasing or diminishing its root by a small quantity.

Thus, if $\phi(x)$ be any function of (x), and (h) a small quantity representing the increment of (x), then, if there be any value of (x), such that $\phi(x+h)$ and $\phi(x-h)$ are both less than $\phi(x)$, $\phi(x)$ is a maximum.

Def. A function is said to be at its minimum value, when its value is increased either by increasing or diminishing the root by a small quantity.

Thus if $\phi(x+h)$ and $\phi(x-h)$ be both greater than $\phi(x)$, $\phi(x)$ is a minimum.

If we suppose $\phi(x) =$ the sine of an arc, whose versed sine is (x), then while (x) increases from 0 to 1, $\phi(x)$ also increases from 0 to 1, and while (x) increases from 1 to 2, $\phi(x)$ decreases from 1 to 0; here then $\phi(x)$ is at its maximum value, when $x=1$, for it appears, that if we suppose (x) to be either greater or less than unity, $\phi(x)$ is in each case less than unity.

A function may have more than one maximum or minimum value. Suppose

$$\phi(x) = (x-a)\,(x-b)\,(x-c)\,(x-d)\,(x-e),$$

where (a), (b), (c), (d), (e), are taken in order, beginning with the greatest; now it is manifest, that if (a) or (b) be substituted for (x), the value of $\phi(x)$ becomes (0); and if a quantity lying between (a) and (b) be substituted, the value of $\phi(x)$ is negative; hence it is manifest, that there must be some quantity lying between (a) and (b), which, if substituted for (x), will give $\phi(x)$ a greater negative value than any other quantity lying between (a) and (b); let this quantity be (α), then $\phi(\alpha)$ is a minimum.

Again, if (b) or (c) be substituted for (x), $\phi(x)$ becomes $=0$, and if a quantity lying between (b) and (c) be substituted for (x), $\phi(x)$ is positive; hence, as before, there is manifestly a quantity lying between (b) and (c), which, substituted for (x), will give a maximum positive result; let this quantity be (β), then $\phi(\beta)$ is a maximum.

In the same manner it may be shewn, that there is a quantity (γ) lying between (c) and (d) such, that

$\phi(\gamma)$ is a minimum, and (δ) lying between (d) and (e) such, that $\phi(\delta)$ is a maximum.

From this then it appears, that (ϕx) has two maximum and two minimum values.

Since $\phi(a)$ is the least value, which $\phi(x)$ can assume, while (x) lies between (a) and (b), if (h) be so small, that both $(a+h)$ and $(a-h)$ lie between (a) and (b), then $\phi(a)$ is less than either $\phi(a+h)$ or $\phi(a-h)$, if, however, (h) be assumed so great, that this condition is not fulfilled, then it does not necessarily follow, that $\phi(a)$ is less than either $\phi(a+h)$ or $\phi(a-h)$, but still $\phi(a)$ is a minimum, according to the definition.

The same remark may be extended to the other maxima and minima values, and from it, it appears, that it is sufficient to constitute $\phi(x)$ a maximum or minimum, if (h) can be assumed so small, that $\phi(x+h)$ or $\phi(x-h)$ are both greater or both less than $\phi(x)$*.

Prop. XXVIII.

When a function is at one of its maxima or minima values, its differential coefficient is equal to nothing.

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* It may be necessary to observe, that of two negative quantities, the one which is less in magnitude, is termed the greater. Thus, if (a) be less than (b), $(-a)$ is said to be greater than $(-b)$; thus also (0) is said to be greater than (-1), (-1) than (-2), and so on. The reason is manifest. If we were to say, for instance, that (-1) is less than (-2), then adding (3) to each side, we should have $(3-1)$, or 2 less than $(3-2)$ or 1. In order to prevent similar inconsistencies from being introduced by the application of the common rules of addition and subtraction, where their absurdity would not be so easily detected, negative quantities are, in the language of Algebra, said to be less in value, when they are greater in magnitude.

Let $\phi(x)$ represent any function of (x);

then $\phi(x+h)=\phi(x)+f\phi(x)\,h+f^2\phi(x)\cdot\frac{h^2}{1.2}+\&c.,$

and $\phi(x)=\phi(x),$

and $\phi(x-h)=\phi(x)-f\phi(x)\,h+f^2\phi(x)\frac{h^2}{1.2}-\&c.$

Now if $f.\phi(x)$ be a finite positive quantity, then, in the series

$$f\cdot\phi(x)\,h+f^2\phi(x)\frac{h^2}{1.2}+\&c.$$

(h) may be made so small, that the first term will be greater than the sum of all the following terms, and in that case, the value of the whole series is positive; and, therefore, $\phi(x+h)$ must be greater than $\phi(x)$.

Again, in the series

$$-f\phi(x)\,h+f^2\phi(x)\frac{h^2}{1.2}-\&c.$$

(h) may be so diminished, that the sum of the series will be altogether negative; and, therefore, $\phi(x-h)$ must be less than $\phi(x)$, that is, if $f\phi(x)$ be a positive finite quantity, $\phi(x)$ is less than $\phi(x+h)$, and greater than $\phi(x-h)$; and, therefore, it is not a maximum or minimum.

In the same manner it may be shewn, that, if $f\phi(x)$ be a negative finite quantity, $\phi(x)$ is neither a maximum nor a minimum. $\phi(x)$, therefore, can only be a maximum or minimum, when $f\phi(x)=0$.

Cor. 1. If the first differential coefficient vanishes, the function will be a maximum, when the second differential coefficient is negative, and a minimum, when it is positive.

For, in that case, since $f.\phi(x) = 0$, we have

$$\phi(x+h) = \phi(x) + f^2\phi(x).\frac{h^2}{1.2} + f^3\phi(x)\frac{h^3}{1.2.3} + \&c.$$

and

$$\phi(x-h) = \phi(x) + f^2\phi(x)\frac{h^2}{1.2} - f^3\phi(x)\frac{h^3}{1.2.3} + \&c.$$

Now if $f^2.\phi(x)$ be negative, (h) may be made so small, that the sum of the two series,

$$f^2\phi(x)\frac{h^2}{1.2} + f^3\phi(x)\frac{h^3}{1.2.3} + \&c.$$

$$f^2.\phi(x)\frac{h^2}{1.2} - f^3\phi(x)\frac{h^3}{1.2.3} + \&c.$$

will be both negative, and in that case $\phi(x+h)$ and $\phi(x-h)$ will be both less than $\phi(x)$, and, therefore, $\phi(x)$ will be a maximum. In the same manner it might have been shown, that if $f^2\phi(x)$ had been positive, $\phi(x)$ would have been a minimum.

COR. 2. If both the first and second differential coefficients had vanished, the function would have been neither a maximum nor a minimum. For in that case, we should have had

$$\phi(x+h) = \phi(x) + f^3.\phi(x)\frac{h^3}{1.2.3}$$
$$+ f^4\phi(x).\frac{h^4}{1.2.3.4} + \&c.$$

and $$\phi(x-h) = \phi(x) - f^3\phi(x)\frac{h^3}{1.2.3}$$
$$+ f^4\phi(x)\frac{h^4}{1.2.3.4} - \&c.$$

in which series it is manifest, that (h) may be so diminished, that the two quantities $\phi(x+h)$ and $\phi(x-h)$ will be one greater, and one less than $\phi(x)$; and, consequently, $\phi(x)$ is neither a maximum nor a minimum.

Cor. 3. If the number of differential coefficients, which vanish, beginning with the first, be odd, the function will be either a maximum or a minimum, according as the succeeding differential coefficient be negative or positive.

Suppose $f\phi(x), f^2\phi(x), f^3\phi(x) \ldots\ldots f^{2n+1}.\phi(x)$ to become each equal to nothing, then we have

$$\phi(x+h) = \phi(x) + f^{2n+2}.\phi(x).\frac{h^{2n+2}}{1.2.3\ldots(2n+2)}$$

$$+f^{2n+3}.\phi(x).\frac{h^{2n+3}}{1.2.3\ldots.(2n+3)} + \&c.$$

and $$\phi(x-h) = \phi(x) + f^{2n+2}.\phi(x)\frac{h^{2n+2}}{1.2.3\ldots.(2n+2)}$$

$$-f^{2n+3}.\phi(x)\frac{h^{2n+3}}{1.2.3\ldots.(2n+3)} + \&c.$$

Now in these series, (h) may be so diminished, that the second term shall be greater than the sum of all the succeeding terms; in that case, if $f^{2n+2}.\phi(x)$ be a negative quantity, both $\phi(x+h)$ and $\phi(x-h)$ will be less than $\phi(x)$, and, therefore, $\phi(x)$ will be a maximum, and if $f^{2n+2}.\phi(x)$ had been positive, $\phi(x)$ would have been a minimum.

Cor. 4. If the number of differential coefficients, which vanish, beginning with the first, be even, the function is neither a maximum nor a minimum.

Suppose $f\phi(x), f^2.\phi(x), f^3.\phi(x) \ldots\ldots f^{2n}.\phi(x)$, to become each equal to nothing, then we have

$$\phi(x+h)=\phi(x)+f^{2n+1}.\phi(x)\frac{h^{2n+1}}{1.2.3\ldots.(2n+1)}$$

$$+f^{2n+2}.\phi(x)\frac{h^{2n+2}}{1.2.3\ldots.(2n+2)}+\text{\&c., and}$$

$$\phi(x-h)=\phi(x)-f^{2n+1}.\phi(x)\frac{h^{2n+1}}{1.2.3\ldots.(2n+1)}$$

$$+f^{2n+2}.\phi(x)\frac{h^{2n+2}}{1.2.3\ldots.(2n+2)}+\text{\&c.}$$

and in these series it appears, that, if we diminish (h) so as to make the second term greater than the sum of all the succeeding terms, one of the two quantities $\phi(x+h)$, and $\phi(x-h)$ will be greater, and one less than $\phi(x)$, and, therefore, $\phi(x)$ will be neither a maximum nor a minimum.

PROP. XXIX.

If ϕ (x) *and* ϕ' (x) *be two functions of* (x), *which both vanish when* x = a, *then the value, which the fraction* $\frac{\phi(\mathrm{x})}{\phi'(\mathrm{x})}$ *assumes, when* (x) *is made* = a, *is* $\frac{\mathrm{f}\phi.(\mathrm{a})}{\mathrm{f}\phi'.(\mathrm{a})}$.

In general $\frac{\phi(x+h)}{\phi'(x+h)}$

$$=\frac{\phi(x)+f\phi(x).h+f^2\phi(x).\frac{h^2}{1.2}+\text{\&c.}}{\phi'(x)+f\phi'(x)h+f^2\phi'(x).\frac{h^2}{1.2}+\text{\&c.}}.$$

Now, making $x=a$, we have, since $\phi(a)=0$, and $\phi'(a)=0$,

$$\frac{\phi(a+h)}{\phi'(a+h)}=\frac{f\phi(a).h+f^2\phi(a).\frac{h^2}{1.2}+\&c.}{f\phi'(a).h+f^2.\phi'(a)\,\frac{h^2}{1.2}+\&c.}$$

$$=\frac{f\phi(a)+f^2\phi(a).\frac{h}{1.2}+\&c.}{f\phi'(a)+f^2.\phi'(a)\,\frac{h^2}{1.2}+\&c.};$$

and making $(h)=0$, we have

$$\frac{\phi(a)}{\phi'(a)}=\frac{f.\phi(a)}{f.\phi'(a)}.$$

Cor. 1. If, when $x=a$, $\phi(x)$, $\phi'(x)$. $f\phi(x)$ and $f\phi'(x)$ all vanish, the value, which the fraction $\frac{\phi(x)}{\phi'(x)}$ assumes when (x) is made $=a$, is $\frac{f^2\phi(a)}{f^2\phi'(a)}$.

Cor. 2. If, when $x=a$,

$$\phi(x), f.\phi(x), f^2\phi(x)\ldots\ldots f^n.\phi(x),$$

become each equal to nothing, and also

$$\phi'(x), f.\phi'(x), f^2\phi(x)\ldots\ldots f^n.\phi'(x),$$

become each equal to nothing, then the value of the fraction $\frac{\phi(x)}{\phi'(x)}$ when $x=a$, becomes $\frac{f^{n+1}.\phi(x)}{f^{n+1}.\phi'(a)}$. For nearly the same demonstration will apply.

SECT. II.

ON THE DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

Prop. XXX.

If an algebraic quantity be raised to any power, its differential is found by multiplying it by its index, diminishing its index by unity, and multiplying by the differential of its root.

Case 1. Let the root be a simple algebraic quantity (x); and let (n) be the power to which it is raised; then

$$d \, . \, x^n = n x^{n-1} dx.$$

$$\text{Let } u = x^n,$$

$$\text{then } u' = (x+h)^n = x^n + n x^{n-1} h$$

$$+ \frac{n \, . \, (n-1)}{1 \, . \, 2} \, . \, x^{n-2} h^2 + \&c.$$

$$\text{but } u' = u + \frac{du}{dx} \, . \, h + \frac{d^2 u}{du^2} \, . \, \frac{h^2}{1 \, . \, 2} + \&c.$$

Where, equating the coefficients of (h), we have

$$\frac{du}{dx} = n x^{n-1}, \quad \text{or } du = n x^{n-1} dx.$$

This proof, since the binomial theorem has been generally proved, will hold good, whether (n) be positive or negative, integer or fractional*.

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* The following demonstration of the binomial theorem, when the index is fractional or negative, will, I hope, appear satisfactory.

$$\text{Assume } X = 1 + \frac{m}{n}x + \frac{\frac{m}{n}\cdot\left(\frac{m}{n}-1\right)}{1\,.\,2}x^2 + \&\text{c.}$$

$$\text{then } X^n = \left\{1 + \left(\frac{m}{n}x + \frac{\frac{m}{n}\cdot\left(\frac{m}{n}-1\right)}{1\,.\,2}x^2 + \&\text{c.}\right)\right\}^n$$

$$= 1 + n\left\{\frac{m}{n}x + \frac{\frac{m}{n}\cdot\left(\frac{m}{n}-1\right)}{1\,.\,2}x^2 + \&\text{c.}\right\}$$

$$+ \frac{n.(n-1)}{1\,.\,2}\cdot\left\{\frac{m}{n}x + \frac{\frac{m}{n}\cdot\left(\frac{m}{n}-1\right)}{1\,.\,2}x^2 + \&\text{c.}\right\}^2$$

$$+ \&\text{c}$$

$$= 1 + Ax + Bx^2 + Cx^3 + \&\text{c.}$$

where A, B, C are the coefficients of (x), (x^2), (x^3), &c. which would have been found by actual multiplication and expansion. Now it is manifest, that the form of each of these coefficients is independent of the numerical values of (m) and (n), and will, therefore, remain the same, whether $\left(\frac{m}{n}\right)$ be an integer or a fraction. First, let the numerical values of (m) and (n) be such, that $\left(\frac{m}{n}\right)$ is a whole number, then, by the first part of the binomial theorem,

$$X = (1+x)^{\frac{m}{n}};$$

and, consequently, $X^n = (1+x)^m \qquad = 1 + mx$

Ex. 1. Hence,

$$d.(a^2+x^2)=d.x^2, \quad \text{(by Prop. 25.)}$$

$$=2x\,dx \quad \text{(by this Prop.)}$$

Ex. 2. $d.(a+bx^4)=d.bx^4$ (by Prop. 25.)

$$=bd.x^4 \quad \text{(by Prop. 23.)}$$

$$=b\times 4x^3dx=4bx^3dx.$$

$$=1+mx+\frac{m.(m-1)}{1.2}x^2+\&c.$$

Hence, in this case, $A=m$, $B=\frac{m.(m-1)}{1.2}$,

and $C=\frac{m.(m-1).(m-2)}{1.2.3}$.

And, since this is true, when $\left(\frac{m}{n}\right)$ is a whole number, it will also be true when $\left(\frac{m}{n}\right)$ is a fraction; therefore it is generally true, that

$$X^n=1+mx+\frac{m.(m-1)}{1.2}x^2+\&c.$$

and, therefore, $X^n=(1+x)^m$;

$$\therefore\ X=(1+x)^{\frac{m}{n}},$$

$$\text{or } (1+x)^{\frac{m}{n}}=1+\frac{m}{n}x+\frac{\frac{m}{n}.\left(\frac{m}{n}-1\right)}{1.2}x^2+\&c.$$

The same form will hold true, if the index be negative.

Let $(-k)$ be the index, and let (m) and (n) be two numbers, such, that $n-m=k$, or $m-n=-k$;

then

Ex. 3. $$d.\left(\frac{a}{x}\right) = d.ax^{-1} = a \times -1$$

$$\times x^{-2} \times dx = -\frac{adx}{x^2}.$$

Ex. 4. $$d.\left(\frac{x^2-a^2}{x^2}\right) = d.\left(1-\frac{a^2}{x^2}\right)$$

$$= d.\left(-\frac{a^2}{x^2}\right) = \frac{2a^2dx}{x^3}.$$

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then, $\overline{1+x}\rvert^{-k} = \overline{1+x}\rvert^{m-n} = \frac{(1+x)^m}{(1+x)^n} = \frac{1+mx+\&c.}{1+nx+\&c.}$

$$= 1+Ax+Bx^2+Cx^3+\&c.$$

where A, B, C, &c. are the coefficients, whose value would have been found by actual division. Now the form of these coefficients is independent of the values of (m) and (n), and will, therefore, remain the same, whether (m) be greater than (n), or (n) than (m). Now, if (m) had been greater than (n), ($m-n$) would have been positive, and, consequently, we should have had

$$\frac{1+mx+\&c.}{1+nx+\&c.} = \frac{(1+x)^m}{(1+x)^n} = (1+x)^{m-n}$$

$$= 1+(m-n)\,x$$

$$+\frac{(m-n)\,(m-n-1)}{1.2}\,x^2 + \&c.$$

The same form will, therefore, be true, when (n) is greater than (m); hence we have, [writing $-k$ for ($m-n$)],

$$(1+x)^{-k} = 1-kx+\frac{(-k)\,(-k-1)}{1.2}\,x^2+\&c.$$

$$= 1-kx+\frac{k\,(k+1)}{1.2}\,x^2-\&c.$$

which form will hold good, whether (k) is a whole number or a fraction.

Ex. 5. $$d.\sqrt{x}=d.x^{\frac{1}{2}}=\frac{1}{2}.x^{\frac{1}{2}-1}dx$$

$$=\frac{1}{2}x^{-\frac{1}{2}}dx=\frac{dx}{2\sqrt{x}}.$$

Ex. 6. $$d.\frac{7x^{\frac{2}{7}}}{2}=\frac{7}{2}\times\frac{2}{7}\times x^{\frac{2}{7}-1}\times dx=\frac{dx}{x^{\frac{5}{7}}}.$$

Ex. 7. $$d.\left\{a+b\sqrt{x}+cx^{3}+\frac{1}{x}\right\}$$

$$=\left\{\frac{b}{2\sqrt{x}}+3cx^{2}-\frac{1}{x^{2}}\right\}dx.$$

Ex. 8. $$d.\left(\frac{x}{y}\right)=d.xy^{-1}=y^{-1}dx+xdy^{-1}$$

$$=\frac{dx}{y}-\frac{xdy}{y^{2}}$$

$$=\frac{ydx-xdy}{y^{2}}.$$

Ex. 9. $$d.x^{2}y^{3}=2y^{3}xdx+3x^{2}y^{2}dy.$$

Case 2. Suppose the root to be a binomial $(a+bx^{r})$, one of whose terms (a) is a constant quantity, and let (n) be the power to which it is raised, then

$$d\,(a+bx^{r})^{n}=n.(a+bx^{r})^{n-1}\times rbx^{r-1}dx.$$

Let $u=(a+bx^{r})^{n}$, and $y=a+bx^{r}$;

then $u=y^{n}$, and $\therefore\ \frac{du}{dy}=ny^{n-1}=n.(a+bx^{r})^{n-1}$;

also $\frac{dy}{dx}=rbx^{r-1}$.

Here then, there are three quantities, u, y, and x, and $\frac{du}{dy}$, the differential coefficient of (u) considered as a function of (y), is equal to $n.(a+bx^r)^{n-1}$, and $\frac{dy}{dx}$ the differential coefficient of (y) considered as a function of (x), is rbx^{r-1}, and, therefore, $\frac{du}{dx}$ the differential coefficient of (u) considered as a function of (x) is, (by Prop. 21.)

$$= n\,(a+bx^r)^{n-1} \times rbx^{r-1},$$

and, therefore,

$$du = n\,.\,(a+bx^r)^{n-1} \times rbx^{r-1}dx.$$

Ex. 1. $d.(a+x)^5 \quad = 5\,.\,(a+x)^4.dx.$

Ex. 2. $d.\sqrt{a^2-x^2} \quad = d.(a^2-x^2)^{\frac{1}{2}}$

$$= \frac{1}{2}.(a^2-x^2)^{-\frac{1}{2}} \times -2xdx = \frac{-xdx}{\sqrt{a^2-x^2}}.$$

Ex. 3. $d.\left(\frac{1}{\sqrt{a^2-x^2}}\right) = d.(a^2-x^2)^{-\frac{1}{2}}$

$$= -\frac{1}{2}.(a^2-x^2)^{-\frac{3}{2}} \times -2xdx = \frac{xdx}{(a^2-x^2)^{\frac{3}{2}}}.$$

Ex. 4. $d.x\sqrt{a^2+x^2}$

$$= dx.\sqrt{a^2+x^2}+x.d.\sqrt{a^2+x^2}$$

$$= dx.\sqrt{a^2+x^2}+\frac{x^2dx}{\sqrt{a^2+x^2}}$$

$$= dx\left\{\frac{a^2+x^2+x^2}{\sqrt{a^2+x^2}}\right\}$$

$$= dx\left\{\frac{a^2+2x^2}{\sqrt{a^2+x^2}}\right\}.$$

Ex. 5. $d.x\sqrt{a^2-x^2}=dx\left\{\frac{a^2-2x^2}{\sqrt{a^2-x^2}}\right\}.$

Ex. 6. $d.\sqrt{a^2+x^2}\times\sqrt{b^2-x^2}$

$$=\sqrt{b^2-x^2}.\times d.\sqrt{a^2+x^2}+\sqrt{a^2+x^2}.d\sqrt{b^2-x^2}$$

$$=\sqrt{b^2-x^2}.\frac{xdx}{\sqrt{a^2+x^2}}-\sqrt{a^2+x^2}.\frac{xdx}{\sqrt{b^2-x^2}}$$

$$=dx\left\{\frac{(b^2-x^2)\,x-(a^2+x^2).\,x}{\sqrt{a^2+x^2}.\sqrt{b^2-x^2}}\right\}$$

$$=dx\left\{\frac{(b^2-a^2)\,x-2x^3}{\sqrt{a^2+x^2}.\sqrt{b^2-x^2}}\right\}.$$

Ex. 7. $d\left(\frac{\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}}\right)$

$$=d.(a^2+x^2)^{\frac{1}{2}}\times(a^2-x^2)^{-\frac{1}{2}}$$

$$=(a^2-x^2)^{-\frac{1}{2}}.d.(a^2+x^2)^{\frac{1}{2}}+(a^2+x^2)^{\frac{1}{2}}\times d.(a^2-x^2)^{-\frac{1}{2}}$$

$$=\frac{xdx}{\sqrt{a^2-x^2}.\sqrt{a^2+x^2}}+\frac{\sqrt{a^2+x^2}.\,xdx}{(a^2-x^2)^{\frac{3}{2}}}$$

$$=dx\left\{\frac{x.(a^2-x^2)+x.(a^2+x^2)}{(a^2-x^2)^{\frac{3}{2}}.\sqrt{a^2-x^2}}\right\}$$

$$=\frac{2a^2xdx}{(a^2-x^2)^{\frac{3}{2}}\sqrt{a^2+x^2}}.$$

Case 3. Next the root be a multinomial of the form

$$a+bx^r+cx^s+\&c.$$

where (r), (s), &c. are either positive, or negative, integer or fractional numbers, and let (n) be the power to which this multinomial is raised, then

$$d\,.\,(a+bx^r+cx^s+\&c.)^n$$
$$=(a+bx^r+cx^s+\&c.)^{n-1}\times(rbx^{r-1}+scx^{s-1}+\&c.)\,dx.$$

For, assuming $u=(a+bx^r+cx^s+\&c.)^n$,

and $y=a+bx^r+cx^s+\&c.$

We may prove, as before, that

$$du=(a+bx^r+cx^s+\&c.)^{n-1},$$
$$(rbx^{r-1}+scx^{s-1}+\&c.)\,.\,dx.$$

Ex. 1. $d\,.\sqrt{2ax-x^2}$

$$=\frac{1}{2}\,.\,(2ax-x^2)^{-\frac{1}{2}}.\,(2a-2x)\,dx$$

$$=\frac{(a-x)\,dx}{\sqrt{2ax-x^2}}.$$

Ex. 2. $d\,.\,(2ax-x)^{\frac{3}{2}}$

$$=\frac{3}{2}\,.\,(2ax-x^2)^{\frac{1}{2}}.\,(2a-2x)\,dx$$

$$=3\sqrt{2ax-x^2}.\,(a-x)\,dx.$$

Ex. 3. $d\,.\sqrt{a+c\sqrt{x}+bx\sqrt{x}}$

$$=d\,.\sqrt{a+cx^{\frac{1}{2}}+bx^{\frac{3}{2}}}$$

$$=\frac{1}{2}\,.\,(a+cx^{\frac{1}{2}}+bx^{\frac{3}{2}})^{-\frac{1}{2}}\times\left(\frac{c}{2\sqrt{x}}+\frac{3b\sqrt{x}}{2}\right)dx$$

$$=\frac{\left(\frac{c}{2\sqrt{x}}+\frac{3b\sqrt{x}}{2}\right)dx}{2\sqrt{a+c\sqrt{x}+bx\sqrt{x}}}$$

$$=\frac{(c+3bx)\,dx}{4\sqrt{ax+cx\sqrt{x}+bx^2\sqrt{x}}}.$$

Ex. 4. $d.x\sqrt{2ax-x^2}$

$$= dx\sqrt{2ax-x^2}+x.d\sqrt{2ax-x^2}$$

$$= dx.\sqrt{2ax-x^2}+\frac{(a-x)\,x.dx}{\sqrt{2ax-x^2}}$$

$$= dx\left\{\frac{2ax-x^2+ax-x^2}{\sqrt{2ax-x^2}}\right\}$$

$$= \frac{(3ax-2x^2)\,dx}{\sqrt{2ax-x^2}}.$$

By the assistance of this proposition, the differentials of all algebraic quantities may be found. Thus

Ex. 1. $d.\sqrt{a^2+\sqrt{a^2-x^2}}=d.\sqrt{a^2+y^2}$,

$$(\text{if } \sqrt{a^2-x^2}=y^2)=\frac{ydy}{\sqrt{a^2+y^2}};$$

$$\text{but } 2ydy=\frac{-xdx}{\sqrt{a^2-x^2}};$$

$$\therefore\ d.\sqrt{a^2+\sqrt{a^2-x^2}}$$

$$=-\frac{xdx}{2\sqrt{a^2-x^2}}\left\{\frac{1}{\sqrt{a^2+\sqrt{a^2-x^2}}}\right\}.$$

Ex. 2. $d.(x^2+\sqrt{x^2-a^2})=dx.\left\{2x+\frac{x}{\sqrt{x^2-a^2}}\right\}.$ (See Cor. 3. Prop. 18.)

The differentiation of algebraic quantities is useful in finding their maxima and minima values.

Ex. 1. To divide a line into two parts, such that the rectangle contained by the two parts is a maximum.

Let (a) = the length of the line.

(x) = one part.

and $(a-x)$ will, therefore, equal the other.

then $x \,.\, (a-x)$ or $ax-x^2$ = (by Prop. 4.) the area of the rectangle.

$$\text{Let } u = ax - x^2;$$

$$\therefore \frac{du}{dx} = a - 2x, \text{ and } \frac{d^2u}{dx^2} = -2.$$

Hence, when $(a-2x)=0$, or $x = \frac{a}{2}$, $\frac{du}{dx} = 0$, and $\frac{d^2u}{dx^2}$ is negative; and $\therefore$ (by Prop. 28.) the area of the rectangle is a maximum.

Ex. 2. To divide a straight line into two parts, so that the sum of the squares of those parts is a minimum.

Let (a) be the given line, (x) and $(a-x)$ the two parts, then the sum of the squares of those parts is $a^2-2ax+2x^2$.

$$\text{Let } u = a^2 - 2ax + 2x^2;$$

$$\therefore \frac{du}{dx} = -2a + 4x,$$

$$\text{and } \frac{d^2u}{dx^2} = 4.$$

Hence, when $(-2a+4x)=0$, or $x = \frac{a}{2}$, $\frac{du}{dx} = 0$, and $\frac{d^2u}{dx^2}$ is positive; and $\therefore$ (by Prop. 28.) the sum of the squares of the two parts (x) and $(a-x)$ is a minimum.

Ex. 3. If $\phi(x)=x^n-px^{n-1}+qx^{n-2}-\&c.$ the roots of the limiting equation

$$nx^{n-1}-(n-1)px^{n-2}+(n-2)qx^{n-3}-\&c.=0,$$

when substituted for (x) in the expression for $\phi(x)$, will make $\phi(x)$ either a maximum or a minimum.

$$\text{Since } \phi(x)=x^n-px^{n-1}+qx^{n-2}-\&c.$$

$$\frac{d.\phi(x)}{dx} \text{ or } f\phi(x)=nx^{n-1}-(n-1)px^{n-2}+\&c.$$

$$\text{and } \frac{d^2\phi(x)}{dx^2} \text{ or } f^2\phi(x)=n.(n-1)x^{n-2}$$

$$-(n-1)(n-2)px^{n-3}+\&c.$$

where the equation $f^2\phi(x)=0$ is the limiting equation to $f\phi(x)=0$.

Let (α), (β), (γ), &c. be the roots of the equation $f\phi(x)=0$, then if (α) be substituted for (x) in the expressions for $\phi(x)$, $f\phi(x)$ and $f^2\phi(x)$, $f\phi(\alpha)=0$, and, therefore, $\phi(\alpha)$ will be a maximum or a minimum according as $f^2\phi(\alpha)$ is a negative or positive quantity.

The same may be proved of all the other quantities (β), (γ), &c.

Cor. 1. If the roots of the equation $\phi(x)=0$, be all possible, then the roots of the successive limiting equatious, $f\phi(x)=0$, $f^2.\phi(x)=0$, &c. will also be all possible. Now, if (α'), (β'), (γ'), &c. be the roots of the equation $f^2\phi(x)=0$, and (α), (β), (γ), &c. the roots of the equation $f\phi(x)=0$, taken in order, beginning with the greatest, then the order of these quantities beginning with the greatest will be

$$(\alpha),\ (\alpha'),\ (\beta),\ (\beta'),\ (\gamma),\ (\gamma'),\ \&c.$$

consequently, if (a) be substituted for (x), $f^2\phi(a)$ will be positive, and, therefore, $\phi(a)$ a minimum. If (β) had been substituted for (x), $f^2\phi(\beta)$ would have been negative; and, therefore, $\phi(\beta)$ a maximum. The same may be proved of the remaining roots.

Cor. 2. If two roots of the equation $f\phi(x)=0$, be equal $\phi(x)$ is neither a maximum nor a minimum. For if the equation $f\phi(x)=0$, has two roots each $=a$, the equation $f^2\phi(x)=0$, has one root $=a$, and, consequently, both $f.\phi(a)=0$, and $f^2\phi(a)=0$; and, therefore, $\phi(a)$ is not a maximum or a minimum.

Cor. 3. If the equation $f\phi(x)=0$, has three roots equal to (a), then the equation $f^2.\phi(x)=0$, has two roots equal to (a), and the equation $f^3\phi(x)=0$, has one root equal to (a), and, therefore, $\phi(a)$ is a maximum or a minimum, according as $f^4.\phi(a)$ is a negative or a positive quantity.

Differentiation is also useful in determining the values of vanishing fractions.

To find the value of $\frac{x^3-a^3}{x^2-a^2}$, when $x=a$.

If we suppose $\phi x=x^3-a^3$, and $\phi'(x)=x^2-a^2$, we have seen, (Prop. 29.) that the value of $\frac{\phi(x)}{\phi'(x)}$, when $x=a$, and, therefore, both $\phi(x)$ and $\phi'(x)=0$, is $\frac{f\phi(a)}{f\phi'(a)}$, but $f.\phi(x)$, or $\frac{d\phi x}{dx}=3x^2$, and $f\phi'(x)=2x$; hence

$$\frac{f\phi(a)}{f\phi'(a)}=\frac{3a^2}{2a}=\frac{3a}{2}.$$

Hence, when $x=a$, $\frac{x^3-a^3}{x^2-a^2}=\frac{3a}{2}$.

Ex. 2. To find the value of $\frac{\sqrt{a}-\sqrt{2a-x}}{\sqrt{a+x}-\sqrt{2a}}$, when $x=a$.

$$\text{Let } \phi x = a - \sqrt{2a-x},$$

$$\text{and } \phi'(x) = \sqrt{a+x} - \sqrt{2a};$$

$$\text{then } \frac{d\phi(x)}{dx} \text{ or } f\phi(x) = \frac{1}{2\sqrt{2a-x}};$$

$$\text{and } \therefore f\phi(a) = \frac{1}{2\sqrt{a}},$$

$$\text{and } \frac{d\phi'(x)}{dx} \text{ or } f.\phi'(x) = \frac{1}{2\sqrt{a+x}};$$

$$\text{and } \therefore f\phi'(a) = \frac{1}{2\sqrt{2a}};$$

$$\text{hence, } \frac{f\phi(a)}{f\phi'(a)} \left(\text{or value of } \frac{\sqrt{a}-\sqrt{2a-x}}{\sqrt{a+x}-\sqrt{2a}}\right.$$

$$\left.\text{when } x=a\right) = \frac{2\sqrt{2a}}{2\sqrt{a}} = \sqrt{2}.$$

PROP. XXXI.

If x^n *be considered as a function of* (x), *its* n$^{\text{th}}$ *differential coefficient is a constant quantity,* (n) *being a positive integer.*

For if $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$, $\frac{d^2y}{dx^2} = n.(n-1)\,x^{n-2}$,

$$\frac{d^3y}{dx^3} = n.(n-1)\,(n-2)\,x^{n-3},$$

$$\text{and } \frac{d^4y}{dx^4} = n.(n-1)\,(n-2)\,(n-3)\,x^{n-4}.$$

And thus by successive differentiations we shall at length arrive at

$$\frac{d^n y}{dx^n} = n.(n-1)\,(n-2)\ldots\ldots\{n-(n-1)\}.x^{n-n}$$

$$= 1.2.3\ldots\ldots(n-1).n,$$

$$\text{because } x^{n-n} = \frac{x^n}{x^n} = 1.$$

Ex. 1. If $y=x^4$, we shall, after four differentiations, arrive at

$$\frac{d^4.y}{dx^4} = 4.3.2.1.x^{4-4} = 1.2.3.4.$$

Ex. 2. If $y=x^5$, we shall, after five differentiations, find that

$$\frac{d^5.y}{dx^5} = 1.2.3.4.5.$$

Cor. 1. Hence, all the differential coefficients after the (n^{th}) vanish, (by Prop. 19.).

Cor. 2. Hence, $(x+h)^n$ may be represented by a series, the number of whose terms is finite, and $=n+1$.

Cor. 3. If (n) had been fractional or negative, none of the differential coefficients would have become constant; and, consequently, the series $(x+h)^n$ would have required a series of an infinite number of terms to express it.

Cor. 4. If (n) be negative, we have

$$(x+h)^n = x^n + \frac{d.x^n}{dx}.h + \frac{d^2.x^n}{dx^2}.\frac{h^2}{1.2} + \&c.$$

but when $x=0$, x^n becomes infinite, and ∴ in this case, Taylor's Theorem fails. (Prop. 18. Cor. 2.) If (n) be fractional, it is manifest, that, since the index of (x) in each of the successive differential coefficients decreases by unity, it must at length become negative; and, consequently, that, when $x=0$, that differential coefficient becomes infinite, and, therefore, in this case, Taylor's theorem fails.

Cor. 5. If $\phi(x)$ be any rational function of (x), in which (m) is the highest power of (x), then $\frac{d^m\phi(x)}{dx^m}$ is a constant quantity. For let

$$\phi(x) = ax^m + bx^{m-1} + \ldots . ex + f;$$

$$\therefore \frac{d.\phi(x)}{dx} = max^{m-1} + (m-1)\, bx^{m-2} + \ldots . + e$$

and, by successive differentiations, we arrive at

$$\frac{d^{m-1}.\phi(x)}{dx^{m-1}} = 2.3.4\ldots\ldots m.ax + 1.2.3\ldots\ldots(m-1).b$$

$$\text{and } \frac{d^m.\phi(x)}{dx^m} = 1.2.3.4\ldots\ldots m.a,$$

(m) being, by the supposition a whole number. Hence, $\phi(x+h)$ may be represented by a series of a finite number of terms.

Cor. 6. Hence, if $\phi(x)$ be any rational function of (x), in which the highest power of (x) is less than (n), and if A, B, C,......K be the values of

$$\phi(x), \frac{d.\phi(x)}{dx}, \frac{d^2.\phi(x)}{dx^2} \ldots\ldots \frac{d^{n-1}\phi(x)}{dx^{n-1}}$$

respectively, when $x=a$, then

$$\frac{\phi(x)}{(x-a)^n} = \frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}}$$

$$+ \frac{C}{1.2.(x-a)^{n-2}} + \ldots\ldots + \frac{K}{1.2\ldots(n-1)(x-a)} \quad (A).$$

For, since all the differential coefficients after $\frac{d^{n-1}\phi(x)}{dx^{n-1}}$ must vanish, (Cor. 5. Prop. 31.), we have

$$\phi(x+h) = \phi x + \frac{d.\phi(x)}{dx}.h +$$

$$\frac{d^2\phi(x)}{dx^2}.\frac{h^2}{1.2} + \ldots\ldots + \frac{d^{n-1}\phi(x)}{dx^{n-1}}.\frac{h^{n-1}}{1.2\ldots(n-1)}$$

and, making $x = a$,

$$\phi(a+h) = A + Bh + C.\frac{h^2}{1.2}$$

$$+ \ldots\ldots\ldots\ldots K.\frac{h^{n-1}}{1.2\ldots\ldots(n-1)},$$

and again, supposing $a+h=x$, and $\therefore h = x-a$, we have

$$\phi(x) = A + B.(x-a) + C.\frac{(x-a)^2}{1.2}$$

$$+ \ldots\ldots\ldots\ldots\ldots K.\frac{(x-a)^{n-1}}{1.2\ldots\ldots(n-1)};$$

and, dividing this equation by $(x-a)^n$, we obtain the equation (A). This formula will afterwards be useful in integration.

SECT. III.

ON LOGARITHMS.

If (a) be any constant quantity, and (x) and (y) two other numbers, whose relation is expressed by the equation

$$y = a^x;$$

then (x) is termed the logarithm of (y) to the base (a). It has been shown, (Wood's Algebra, Arts. 383, 384, &c.) that the sum of the logarithms of two numbers is the logarithm of their product, that the difference of their logarithms, is the logarithm of their quotient; and that if the logarithm of a number be multiplied or divided by (n), the result will be the logarithm of the (n^{th}) power or (n^{th}) root of that number.

The calculation of logarithms must, at first, have been a laborious task. They have, however, been calculated and arranged in tables, and the utility of these tables in arithmetical operations is manifest. It would scarcely have been possible to have ascertained the value of such quantities as

$$\overline{7}|^{7.13} \text{ or } \overline{1.05}|^{476},$$

without the assistance of logarithmic tables. Now, however, that these tables have been constructed, we can find the value of such quantities as $(7)^{7.13}$ by the following more simple process. Find, from the tables, the logarithm of (7); multiply the logarithm thus found by 7.13, and the result will be the logarithm of the number required; and, looking again into the tables for the number which answers to this logarithm, we find the value of $\overline{7}|^{7.13}$.

The common logarithms are those, whose base is the number 10. Thus, if $y = 10^x$, (x) is the common logarithm of (y); and if for (y) we substitute, in this equation, the numbers

$$10,\ 100,\ 1000,\ 10000,\ \&c.$$

successively, the corresponding values of (x) will be

$$1,\ 2,\ 3,\ 4,\ \&c.$$

respectively, because $10 = 10^1$, $100 = 10^2$, $1000 = 10^3$, and $10000 = 10^4$, that is, the numbers 1, 2, 3, 4, &c. are the common logarithms of 10, 100, 1000, 10000, respectively.

One reason, why the number 10 is preferred as the base of the common system of logarithms, is the facility it gives us of finding the logarithms of numbers, connected with decimal fractions, from knowing the logarithms of all whole numbers. Thus, if

$$x = \log(275),\ \log(27.5) = \log\left(\frac{275}{10}\right)$$

$$= \log(275) - \log(10) = x - 1.$$

In the same manner it appears, that

$$\log(2.75) = \log\left(\frac{275}{100}\right)$$

$$= \log(275) - \log(100) = x - 2,$$

$$\text{and } \log(.275) = \log\left(\frac{275}{1000}\right)$$

$$= \log(.275) - \log(1000) = x - 3,$$

$$\text{and } \log(.0275) = \log\left(\frac{275}{10000}\right) = x - 4,$$

$$\text{and } \log.(.00275) = \log\left(\frac{275}{100000}\right) = x - 5.$$

From this it is manifest, that in constructing a table of logarithms, whose base is (10), is it necessary to calculate only the logarithms of whole numbers.

There is another system of logarithms called hyperbolic logarithms, which are of great use in the differential and integral calculus. The base of this system is the sum of the series

$$1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \&c.$$

and is generally represented by the Greek letter (ϵ), thus if $y = \epsilon^x$, (x) is the hyperbolic logarithm of (y), and is represented by hyp. log. (y).

If in the equation $y = \epsilon^x$, we substitute (ϵ) for (y), the corresponding value of (x) will be unity, that is, hyp. log (ϵ) = 1.

Prop. XXXII.

If (a) *be any number, whose hyperbolic logarithm is* (A), *and* (x) *its index, then*

$$a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c.$$

For, in general,

$$\phi(x+h)=\phi(x)+f\phi(x).h+f^2\phi(x)\frac{h^2}{1.2}+\&c.$$

Assume $\phi(x)=a^x$, then $\phi(x+h)=a^{x+h}$;

$$\text{hence, } a^{x+h}=a^x+f.(a^x).h+f^2.(a^x).\frac{h^2}{1.2}+\&c.$$

and dividing by a^x, we have

$$a^h=1+\frac{f.(a^x)}{a^x}h+\frac{f^2.(a^x)}{a^x}.\frac{h^2}{1.2}+\&c.$$

Now it is manifest, that the value of (a^h) is independent of the variable (x), and, therefore, the coefficients of (h) in the series representing the value of (a^h), must remain constant, while we suppose (x) to vary.

Assume, therefore, $\frac{f(a^x)}{a^x}=K$, where K is a constant quantity, involving neither (x) nor (h),

$$\text{then } f.(a^x)=K.a^x,$$

and, taking the differential coefficients, we have

$$\begin{aligned} f^2.(a^x) &= f.(K.a^x) \\ &= K.fa^x \text{ (by Prop. 23.)} \\ &= K^2.a^x. \end{aligned}$$

In the same manner it appears, that

$$f^3.(a^x)=K^3.a^x,$$

and so on for the other coefficients. Hence, by substitution, we have

$$a^{x+h}=a^x+Ka^x.\frac{h}{1}+K^2a^x\frac{h^2}{1.2}+\&c.$$

and, dividing by (a^x),

$$a^h = 1 + Kh + \frac{K^2h^2}{1\,.\,2} + \&c. \qquad (M).$$

which is true whatever be the value of (h); let, therefore, $h = \frac{1}{K}$, and we have

$$a^{\frac{1}{K}} = 1 + 1 + \frac{1}{1\,.\,2} + \&c.$$

$$= \epsilon;$$

$\therefore\ a = \epsilon^K$, and, therefore, $K =$ hyp. log. $(a) = A$.

Substituting, therefore, A for K, and (x) for (h), in the series (M), we have

$$a^x = 1 + Ax + \frac{A^2x^2}{1\,.\,2} + \frac{A^3a^3}{1\,.\,2\,.\,3} + \&c.$$

Cor. 1. Hence, $\epsilon^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \&c.$

for, if we suppose (a) to become $= \epsilon$, then (A) must become unity.

Cor. 2. Hence, $d\,.\,a^x = A\,.\,a^x dx$,

$$\text{for } f\,.\,(a^x) \text{ or } \frac{d\,.\,(a^x)}{dx} = A\,.\,a^x.$$

Cor. 3. Hence, $d\,.\,(\epsilon^x) = \epsilon^x dx$.

Cor. 4. Hence, all the differential coefficients of ϵ^x are equal to ϵ^x;

$$\text{for since } d\,(\epsilon^x) = \epsilon^x dx,$$

$$f\,.\,\epsilon^x \text{ or } \frac{d\,.\,(\epsilon^x)}{dx} = \epsilon^x;$$

$$\text{hence, } f'^{2}.\,\epsilon^{x} \text{ or } \frac{d.f(\epsilon^{x})}{d.x} = \frac{d.(\epsilon^{x})}{dx} = \epsilon^{x}.$$

In the same manner it appears, that

$$f^{3}.(\epsilon^{x}) \text{ and } f^{n}(\epsilon^{x}), \text{ \&c. each} = \epsilon^{x}.$$

Cor. 5. The logarithm of unity to any base, is equal to nothing. For if $a^{x} = 1$, we have

$$1 = 1 + Ax + \frac{A^{2}x^{2}}{1.2} + \text{\&c.}$$

$$\text{or } Ax + \frac{A^{2}x^{2}}{1.2} + \text{\&c.}$$

always $= 0$, whatever be the values of A and (a), which cannot be the case, unless (x) always $= 0$.

Prop. XXXIII.

If (y) *be the hyperbolic logarithm of* (1 + x), *then*

$$y = x - \frac{x^{2}}{2} + \frac{x^{3}}{3} - \frac{x^{4}}{4} + \text{\&c.}$$

By Taylor's theorem,

$$\text{hyp. log. } (x+h) = \text{hyp. log. } (x) + f'.\text{ hyp. log. } (x)\frac{h}{1}$$

$$+ f'^{2}.\text{ hyp. log. } (x)\frac{h^{2}}{1.2} + \text{\&c.}$$

Now, if $u = \text{hyp. log. } (x)$, $\epsilon^{u} = x$,

and by the former proposition it appears, that the differential coefficient of (x), considered as a function of $(u) = \epsilon^{u} = x$, hence, the differential coefficient of (u) considered as a function of (x) is $= \frac{1}{x}$, (by Prop. 20.),

$$\text{that is, } f.\text{hyp. log. }(x) = \frac{1}{x};$$

$$\therefore \frac{d.f.\text{hyp. log. }(x)}{dx} \text{ or } f^2.\text{hyp. log. }(x) = \frac{-1}{x^2},$$

$$\text{and } \therefore \frac{d.f^2.\text{hyp. log. }(x)}{dx} \text{ or } f^3.\text{hyp. log. }(x) = \frac{1.2}{x^3};$$

$$\text{and } \therefore \frac{df^3.\text{hyp. log. }(x)}{dx} \text{ or } f^4.\text{hyp. log. }(x) = \frac{-1.2.3}{x^4},$$

and, by substituting these values in the series for hyp. log. $(x+h)$, we have

$$\text{hyp. log. }(x+h) = \text{hyp. log. }(x) + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \&\text{c.}$$

Making $x=1$, we have, since hyp. log. $(1)=0$,

$$\text{hyp. log. }(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \&\text{c.}$$

And, writing (x) for (h), we have

$$\text{hyp. log. }(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&\text{c.} \quad (B).$$

COR. 1. Hence, $d.\text{hyp. log. } x = \frac{dx}{x}$;

and, therefore, the differential of the hyperbolic logarithm of any quantity, is equal to the differential of that quantity, divided by the quantity itself.

Cor. 2. Hence, writing $(-x)$ in series (B), we have

$$\text{hyp. log. } (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \&c. \qquad (C),$$

and, therefore, the hyp. log. of a quantity less than unity is negative.

Cor. 3. Hence, subtracting series (C) from series (B), we have

$$\text{hyp.log. } (1+x) - \text{hyp.log. } (1-x) \text{ or hyp.log. } \left(\frac{1+x}{1-x}\right)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \&c.\right). \qquad (D).$$

Cor. 4. If we assume $x = \frac{y-1}{y+1}$, we have

$$\text{hyp. log. } (y) = 2 \, . \left\{\frac{y-1}{y+1} + \frac{1}{3} \, . \left(\frac{y-1}{y+1}\right)^3 \right.$$

$$\left. + \frac{1}{5}\left(\frac{y-1}{y+1}\right)^5 + \&c.\right\}. \qquad (E).$$

Cor. 5. If we assume $y = \frac{1+z}{z}$, we have

$$\text{hyp. log. } \left(\frac{1+z}{z}\right) \text{ or hyp. log. } (1+z) - \text{hyp. log. } (z)$$

$$= 2\left\{\frac{1}{2z+1} + \frac{1}{3} \, . \, \frac{1}{(2z+1)^3} + \&c.\right\};$$

and, therefore,

$$\text{hyp. log. } (1+z) = \text{hyp. log. } z + 2\left\{\frac{1}{2z+1}\right.$$

$$\left. + \frac{1}{3} \, . \, \frac{1}{(2z+1)^3} + \&c.\right\} \qquad (F).$$

Cor. 6. If, in the series (E), we make $y=2$, we have

$$\text{hyp. log. } (2)=2\left\{\frac{1}{3}+\frac{1}{3 \,.\, 3^3}+\frac{1}{5 \,.\, 3^5}+\&\text{c.}\right\}.$$

From whence hyp. log. (2) may be calculated by a series, which converges rapidly. But in the series (B), if we had made $x=1$, we should have had

$$\text{hyp. log. } (2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\&\text{c.}$$

a series, which converges, indeed, but so slowly, as to be never used in calculation.

Again, if in the series (E), we had made $y=3$, we should have had

$$\text{hyp. log. } (3)=2\left(\frac{1}{2}+\frac{1}{3 \,.\, 2^3}+\frac{1}{5 \,.\, 2^5}+\&\text{c.}\right).$$

whence, hyp. log. (3) might be calculated. But having before calculated hyp. log. (2), we might have used the series (F), and have made $z=2$, whence

$$\text{hyp. log. } (3)=\text{hyp. log. } (2)+2\left\{\frac{1}{5}+\frac{1}{3 \,.\, 5^3}\right.$$
$$\left.+\frac{1}{5 \,.\, 5^5}+\&\text{c.}\right\},$$

a series which converges with greater rapidity than the one above.

Again, hyp.log. (4) = hyp.log. (2^2) = 2.hyp.log. 2.

$$=4\left(\frac{1}{3}+\frac{1}{3 \,.\, 3^3}+\frac{1}{5 \,.\, 3^5}+\&\text{c.}\right).$$

Again, hyp. log. (5) = hyp. log. 4

$$+ 2 \left(\frac{1}{9} + \frac{1}{3 . 9^3} + \&c.\right).$$

as appears by making $z = 4$, in the series (F).

Again, hyp. log. (6) = hyp. log. (2) + hyp. log. (3), both which hyp. logs. have been already calculated.

Again, hyp. log. (7) = hyp. log. (6)

$$\left(\frac{1}{13} + \frac{1}{3 . 13^3} + \&c.\right).$$

hyp. log. (8) = hyp. log. (2^3) = 3 hyp. log. (2),

hyp. log. (9) = hyp. log. (3^2) = 2 hyp. log. (3),

hyp. log. (10) = hyp. log. (5) + hyp. log. (2).

In a similar manner, the hyperbolic logarithms of higher numbers might have been calculated. As, however, it is not the object of academical education to send out into the world a multitude of expert logarithmic calculators, it would be useless to prosecute this subject any further. Those, who have undertaken the laborious task of constructing logarithmic tables, have discovered a great variety of curious artifices, by which they have materially diminished the labour of calculation. A knowledge of these artifices, ingenious as they may be, is, like the knowledge of verbal criticism, useful in its way, but, since it has no tendency either to enlarge the mind, or to strengthen the understanding, it should be left to the pursuit of those, who intend to devote their lives to the study of Mathematics.

PROP. XXXIV.

Having found the hyperbolic logarithms of numbers, to find their logarithms to any other base.

Let (a) be the given base,

(y) the number, whose logarithm is required,

(x) the logarithm required;

then $y=a^x$;

$$\therefore \text{hyp. log. }(y)=\text{hyp. log. }(a^x)=x\,.\,\text{hyp. log. }(a);$$

$$\therefore x=\frac{\text{hyp. log. }(y)}{\text{hyp. log. }(a)}.$$

COR. 1. Hence, the common logarithm of (2), or $\text{log. }(2)=\dfrac{\text{hyp. log. }(2)}{\text{hyp. log. }(10)}$, and having calculated the hyperbolic logarithms of (2) and of (10) by the former proposition, we may, by division, find the common logarithm of (2). In the same way we may find the common logarithm of all other numbers.

COR. 2. Hence, if (a) and (a') be the bases of two systems of logarithms, and (y) any number, then

$$\frac{\text{log. }(y)\text{ to base }(a)}{\text{log. }(y)\text{ to base }(a')}=\frac{\text{hyp. log. }(a')}{\text{hyp. log. }(a)}.$$

For, by this proposition,

$$\text{log. }(y)\text{ to base }(a)=\frac{\text{hyp. log. }(y)}{\text{hyp. log. }(a)};$$

$$\text{and log. }(y)\text{ to base }(a')=\frac{\text{hyp. log. }(y)}{\text{hyp. log. }(a')};$$

hence, by division, we derive the above equation.

From this it appears, that having calculated the logarithms of numbers to any one base, we may easily find the logarithms of the same numbers to any other base.

From the preceding propositions, we may find the means of differentiating functions involving variable indices.

To find the differential of y^z, where (y) and (z) represent some functions of (x).

$$\text{Assume } y^z = w;$$

$$\therefore\ z\,.\,\text{hyp. log. } y = \text{hyp. log. } w;$$

$$\text{hence, hyp. log. } y\,.\,dz + \frac{z\,dy}{y} = \frac{d\,.\,w}{w};$$

$$\therefore\ dw = y^z\left(\text{hyp. log. } y\,.\,dz + \frac{z\,dy}{y}\right).$$

COR. 1. If $y = z = x$, we have

$$d\,.\,x^x = x^x dx\,\{\text{hyp. log. }(x) + 1\}.$$

To find the differential of $x^{\frac{1}{x}}$.

$$\text{Assume } u = x^{\frac{1}{x}};$$

$$\therefore\ \text{hyp. log. } u = \frac{1}{x}\,.\,\text{hyp. log. } x;$$

$$\therefore\ \frac{du}{u} = \frac{-\text{hyp. log. }(x)\,.\,dx}{x^2} + \frac{1}{x}\,.\,\frac{dx}{x}$$

$$= \frac{dx}{x^2}\,.\,\{1 - \text{hyp. log. }(x)\};$$

$$\therefore\ du = \frac{x^{\frac{1}{x}}dx}{x^2}\,.\,\{1 - \text{hyp. log. }(x)\}.$$

Cor. 1. If the value of (x) be such, that $1 -$ hyp. log. $(x) = 0$, it is manifest, that $\frac{du}{dx} = 0$; and, therefore, u is a maximum; but, in this case,

$$\text{hyp. log. } x = 1 = \text{hyp. log. } \epsilon\text{;}\ \therefore\ x = \epsilon.$$

To find the differential of hyp. log. $(1 + x)$, and hyp. log. $(1 - x)$,

$$d\,.\,\text{hyp. log. }(1 + x) = \frac{d\,.\,(1 + x)}{1 + x} = \frac{dx}{1 + x},$$

$$\text{and } d\,.\,\text{hyp. log. }(1 - x) = \frac{d\,.\,(1 - x)}{1 - x} = \frac{-dx}{1 - x}.$$

The following differentials are useful to remember.

$$d\,.\,\text{hyp. log. }(x + \sqrt{x^2 \pm a^2})$$

$$= \frac{d\,.(x + \sqrt{x^2 \pm a^2})}{x + \sqrt{x^2 \pm a^2}}$$

$$= \frac{dx\,.\left(1 + \frac{x}{\sqrt{x^2 \pm a^2}}\right)}{x + \sqrt{x^2 \pm a^2}}$$

$$= dx \frac{(\sqrt{x^2 \pm a^2} + x)}{\sqrt{x^2 \pm a^2}\,.\,(x + \sqrt{x^2 \pm a^2})}$$

$$= \frac{dx}{\sqrt{x^2 \pm a^2}},$$

$$d\,.\,\text{hyp. log. }(x \pm a + \sqrt{x^2 \pm 2ax})$$

$$= \frac{d\,.\,(x \pm a + \sqrt{x^2 \pm 2ax})}{x \pm a + \sqrt{x^2 \pm 2ax}}$$

$$= \frac{dx\,.\left(1 + \frac{x \pm a}{\sqrt{x^2 \pm 2ax}}\right)}{x \pm a + \sqrt{x^2 \pm 2ax}}$$

$$= \frac{dx}{\sqrt{x^2 \pm 2ax}}.$$

$$d\,.\,\text{hyp. log.}\left(\frac{a+x}{a-x}\right)$$

$$= d\,.\,\text{hyp. log.}\,(a+x) - d\,.\,\text{hyp. log.}\,(a-x)$$

$$= \frac{dx}{a+x} + \frac{dx}{a-x}$$

$$= \frac{2a\,dx}{a^2 - x^2},$$

$$d\,.\,\text{hyp. log.}\left(\frac{x-a}{x+a}\right)$$

$$= d\,.\,\text{hyp. log.}\,(x-a) - d\,.\,\text{hyp. log.}\,(x+a)$$

$$= \frac{dx}{x-a} - \frac{dx}{x+a}$$

$$= \frac{2a\,dx}{x^2 - a^2},$$

$$d\,.\,\text{hyp. log.}\left(\frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 + x^2}}\right)$$

$$= d\,.\,\text{h. l.}\,(a - \sqrt{a^2 - x^2}) - d\,.\,\text{h. l.}\,(a + \sqrt{a^2 - x^2})$$

$$= \frac{x\,dx}{\sqrt{a^2 - x^2}.(a - \sqrt{a^2 - x^2})} + \frac{x\,dx}{\sqrt{a^2 - x^2}\,(a + \sqrt{a^2 - x^2})}$$

$$= \frac{2a\,dx}{x\sqrt{a^2 - x^2}},$$

$$d\,.\,\text{hyp. log.}\left(\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}\right)$$

$$=d\,.\,\text{h.l.}\,(\sqrt{a^2+x^2}-a)-d\,.\,\text{h.l.}\,(\sqrt{a^2+x^2}+a)$$

$$=\frac{x\,dx}{\sqrt{a^2+x^2}\,(\sqrt{a^2+x^2}-a)}-\frac{x\,dx}{\sqrt{a^2+x^2}\,(\sqrt{a^2+x^2}+a)}$$

$$=\frac{2a\,dx}{x\sqrt{a^2+x^2}}.$$

These six preceding forms will, hereafter, be useful in integration.

The differentiation of algebraic functions is sometimes rendered easier, by first taking their hyperbolic logarithms, and then differentiating; thus, if

$$u=\sqrt{\frac{a^2+x^2}{a^2-x^2}},$$

$$\text{h. l. } u=\frac{1}{2}\,.\,\text{h. l. }(a^2+x^2)-\frac{1}{2}\,.\,\text{h. l. }(a^2-x^2),$$

$$\therefore\ \frac{du}{u}=\frac{x\,dx}{a^2+x^2}+\frac{x\,dx}{a^2-x^2}$$

$$=\frac{2a^2x\,dx}{a^4-x^4};$$

$$\therefore\ du=\sqrt{\frac{a^2+x^2}{a^2-x^2}}\cdot\frac{2a^2x\,dx}{a^4-x^4}$$

$$=\sqrt{\frac{a^2+x^2}{a^2-x^2}}\cdot\frac{2a^2x\,dx}{(a^2+x^2)\,(a^2-x^2)}$$

$$=\frac{2a^2x\,dx}{(a^2-x^2)^{\frac{3}{2}}\,.\sqrt{a^2+x^2}}.$$

From the preceding propositions, we may also find the values of vanishing fractions, whose numerators and denominators involve variable exponentials.

Ex. 1. To find the value of $\frac{a^x - 1}{x}$, when $x = 0$.

The differential coefficient of the numerator $=$ $A a^x$, and when $x = 0$, it $= A$. Also the differential coefficient of the denominator $= 1$, whatever be the value of x; hence, when $x = 0$, $\frac{a^x - 1}{x} = \frac{A}{1} =$ h. l. (a).

This might have been shown independently of differentiation, for

$$a^x = 1 + Ax + \frac{A^2 x^2}{1 . 2} + \&c.;$$

$$\text{hence, } a^x - 1 = Ax + \frac{A^2 x^2}{1 . 2} + \&c.;$$

$$\therefore \frac{a^x - 1}{x} = A + \frac{A^2 x}{1 . 2} + \&c.;$$

and, consequently, when $x = 0$, $\frac{a^x - 1}{x}$, must $= A =$ hyp. log. (a).

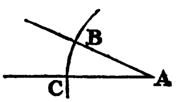
Ex. 2. To find the value of $\frac{\text{hyp. log. } (1 + x)}{x}$, when $x = 0$.

The differential coefficient of hyp. log. $(1 + x)$ $= \frac{1}{1 + x}$, and when $x = 0$, it becomes unity. Also the differential coefficient of (x) always equals unity; hence, the value of $\frac{\text{hyp. log. } (1 + x)}{x}$, when $x = 0$, is unity.

SECT. IV.

ON CIRCULAR FUNCTIONS.

Def. If the vertex of any angle be made the centre of a circle, whose radius is equal to a linear unit, the numerical value of this angle is represented by that number, which expresses the length of the circular arc intercepted between the two straight lines, containing the angle.



Thus, let *BAC* be an angle, *CA* equal a linear unit, and let *CB* be a circular arc, described from centre *A*, with radius *CA*, then if (θ) be the numerical value of the length of the arc *CB*, it will also, by the definition, be the numerical value of the angle *BAC*.

In general, when we call an angle (θ), we mean by (θ) its numerical value, and when we make use of the symbols, $\sin\theta$, $\cos\theta$, and $\tan\theta$, we intend them to represent the sine, cosine, and tangent of the angle (θ), to radius unity.

If (z) be the arc, which the angle (θ) subtends at the centre of a circle, whose radius $= a$; and if $\sin z$, $\cos z$ and $\tan z$, be the sine, cosine, and tangent of the angle to this radius, we have, (by Trigonometry)

$$\sin\theta = \frac{\sin z}{a}, \quad \cos\theta = \frac{\cos z}{a}, \text{ and } \tan\theta = \frac{\tan z}{a}.$$

In the succeeding propositions, we shall take for granted the following well-known axiom, viz. that the length of a circular arc is less than that of its tangent, and greater than that of its chord or sine.

Prop. XXXV.

If (θ) be an angle, the differential coefficient of its sine, considered as a function of (θ), is equal to $\cos\theta$.

For if we suppose (h) to be the increment of (θ), we have, by Trigonometry,

$$\begin{aligned}2\cos\theta.\sin h &= \sin(\theta+h)-\sin(\theta-h)\\ &= \left.\begin{aligned}&\sin\theta+f.\sin\theta.h+f^2.\sin\theta.\frac{h^2}{1.2}+\&c.\\ &-(\sin\theta-f.\sin\theta.h+f^2.\sin\theta.\frac{h^2}{1.2}-\&c.)\end{aligned}\right\}\\ &= 2.f.\sin\theta.h+2.f^3.\sin\theta.\frac{h^3}{1.2.3}+\&c.\end{aligned}$$

Consequently,

$$\sin h=\frac{f.\sin\theta}{\cos\theta}.h+\frac{f^3.\sin\theta}{\cos\theta}.\frac{h^3}{1.2.3}+\&c.\quad (A).$$

Now, since by the axiom we have assumed (h) is greater than $\sin h$, $h-\sin h$, or the series

$$\left(1-\frac{f.\sin\theta}{\cos\theta}\right).h-\frac{f^3.\sin\theta}{\cos\theta}.\frac{h^3}{1.2.3}-\&c.$$

must be positive; and $\therefore \dfrac{f.\sin\theta}{\cos\theta}$ cannot be greater than 1. (For if it were so, the coefficient of the first term in the series would be negative, and, consequently, by diminishing (h), so that the first term becomes greater than the sum of all the rest, we might make the value of the whole series negative, which is absurd).

$$\text{Again, } \tan h = \frac{\sin h}{\cos h} = \frac{\sin h}{\sqrt{1-\sin^2 h}}$$

$$= \sin h \, . \, (1-\sin^2 h)^{-\frac{1}{2}}$$

$$= \sin h + \frac{1}{2} \, . \, \sin^3 h + \&c.$$

$$= \frac{f \, . \, \sin \theta}{\cos \theta} \, . \, h + B \, . \, h^3 + \&c.$$

as will appear by substituting the value of sin h, which was found in the equation (A); but, by the assumed axiom, tan h is greater than (h); hence, tan $h-h$, or the series

$$\left(\frac{f \, . \sin \theta}{\cos \theta} - 1\right) . \, h + B h^3 + \&c.$$

must be positive; and, therefore, $\frac{f \, . \sin \theta}{\cos \theta}$ cannot be less than 1, and, consequently, we have

$$\frac{f \, . \sin \theta}{\cos \theta} = 1, \text{ or } f \, . \sin \theta = \cos \theta.$$

Cor. 1. Hence, $\frac{d \, . \sin \theta}{d\theta} = \cos \theta$,

$$\text{or } d \, . \sin \theta = \cos \theta \, . \, d\theta.$$

Cor. 2. Hence, $d \, . \cos \theta = -\sin \theta \, . \, d\theta$.

For $\sin^2 \theta + \cos^2 \theta = 1$, and differentiating,

$$2\sin\theta\,.\,d.\sin\theta+2\cos\theta\,.\,d\,.\cos\theta=0\,;$$

$$\therefore\ \sin\theta\,.\cos\theta\,.\,d\theta+\ \cos\theta\,.\,d\,.\cos\theta=0\,;$$

$$\therefore\ d\,.\cos\theta=-\sin\theta\,.\,d\theta.$$

COR. 3. Hence, $d\,.\,\text{ver sin}\ \theta=d\,(1-\cos\theta)$

$$=-d\,.\cos\theta$$

$$=\sin\theta\,.\,d\theta.$$

COR. 4. Hence, $d\,.\,\text{sec.}\ \theta=\sec\theta\,.\tan\theta\,.\,d\theta.$

$$\text{For } d\,.\sec\theta=d\,.\frac{1}{\cos\theta}=\frac{-d\,.\cos\theta}{\cos^2\theta}$$

$$=\frac{\sin\theta\,.\,d\theta}{\cos^2\theta}=\frac{\sin\theta}{\cos\theta}\,.\,\frac{d\theta}{\cos\theta}$$

$$=\sec\theta\,.\tan\theta\,.\,d\theta.$$

COR. 5. Hence, $d\,.\tan\theta=\sec^2\theta\,.\,d\theta.$

For $\tan^2\theta=\sec^2\theta+1\,;$

$$\therefore\ \tan\theta\,.\,d\tan\theta=\sec\theta\,.\,d\sec\theta$$

$$=\sec^2\theta\,.\tan\theta\,.\,d\theta\,;$$

$$\therefore\ d\tan\theta=\sec^2\theta\,.\,d\theta.$$

COR. 6. Hence, if (z) be a circular arc, whose radius is (a), and if (y) (x) (x') (s) (t) be its sine, cosine, versed sine, secant, and tangent respectively to radius (a), then we have the following equations:

$$dz=\frac{a\,dy}{\sqrt{a^2-y^2}},\quad dz=\frac{-a\,dx}{\sqrt{a^2-x^2}},\quad dz=\frac{a\,dx'}{\sqrt{2ax'-x'^2}},$$

$$dz=\frac{a^2ds}{s\sqrt{s^2-a^2}},\quad dz=\frac{a^2dt}{a^2+t^2}.$$

For since, by Trigonometry, we have, supposing (k) the increment of (z)

$$2\cos z \,.\, \sin k = a\,\{\sin(z+k) - \sin(z-k)\}$$

$$= \frac{2a\,.\,d\,.\sin z}{dz}\,.\,k + \frac{2a\,.\,d^3.\sin z}{dz^3}\,.\,\frac{k^3}{1\,.\,2\,.\,3} + \&c.$$

We may prove as before, that

$$\frac{a\,.\,d\sin z}{dz\,.\cos z} = 1;$$

$$\text{and } \therefore \frac{dy}{dz} = \frac{\cos z}{a} = \frac{\sqrt{a^2-y^2}}{a}.$$

Thus we have the differential coefficient of (y) considered as a function of $(z) = \frac{\sqrt{a^2-y^2}}{a}$; hence, the differential coefficient of (z) considered as a function of $(y) = \frac{a}{\sqrt{a^2-y^2}}$; (by Prop. 20.)

$$\text{or } \frac{dz}{dy} = \frac{a}{\sqrt{a^2-y^2}};$$

$$\therefore dz = \frac{a\,dy}{\sqrt{a^2-y^2}}.$$

In the same manner the other formulæ may be proved.

Cor. 7. Supposing, as before, $\sin\theta$, and $\cos\theta$, to represent the sine and cosine of the angle (θ) to radius unity, while $\sin z$, and $\cos z$, represents them to radius (a), we have, (by Trigonometry),

$$\sin\theta = \frac{\sin z}{a}, \text{ and } \cos\theta = \frac{\cos z}{a};$$

$$\therefore\ d.\sin\theta = \frac{d.\sin z}{a} = \frac{dz.\cos z}{a^2} = \frac{dz.\cos\theta}{a},$$

$$\text{but } d.\sin\theta = d\theta.\cos\theta;$$

$$\text{hence, } \frac{dz}{a} = d\theta, \text{ or } dz = ad\theta = d.a\theta.$$

Now, since the differentials of (z) and of ($a\theta$) are equal, the difference between these quantities must be constant; but when $z = 0$, θ, and $\therefore$ $a\theta = 0$; hence, in that case, their difference $= 0$, and, therefore, their difference must always $= 0$; hence, (z) must always equal ($a\theta$), and $\therefore$ $\theta = \frac{z}{a}$.

From whence it appears, that when the angle is given, the arc is proportional to the radius.

Cor. 8. If we suppose (θ) to be a right angle, we shall have the quadrantal arcs of different circles proportional to their radii; and, therefore, four times the quadrantal arcs of these circles, or their circumferences will be found to be in proportion to their radii.

Prop. XXXVI.

To express the sin θ and cos θ, in terms of (θ).

We have seen that

$$\sin h = \frac{d.\sin\theta}{d\theta.\cos\theta}\,h + \frac{d^3.\sin\theta}{d\theta^3.\cos\theta}\cdot\frac{h^3}{1.2.3}$$

$$+\frac{d^5.\sin\theta}{d\theta^3.\cos\theta}\cdot\frac{h^5}{1.2.3.4.5} + \&c.$$

whatever be the value of θ;

but $\frac{d.\sin\theta}{d\theta.\cos\theta} = 1$; $\therefore \frac{d.\sin\theta}{d\theta} = \cos\theta$;

$$\therefore \frac{d^2\sin\theta}{d\theta^2} = \frac{d.\cos\theta}{d\theta} = -\sin\theta;$$

hence, $\frac{d^3\sin\theta}{d\theta^3} = \frac{-d.\sin\theta}{d\theta} = -\cos\theta$;

$$\therefore \frac{d^3\sin\theta}{d\theta^3\cos\theta} = -1.$$

In the same manner it may be shewn, that

$$\frac{d^5\sin\theta}{d\theta^5.\cos\theta} = 1.$$

Hence, $\sin h = h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \&c.$

And, substituting (θ) for (h)

$$\sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \&c.$$

Differentiating this equation, and dividing by $d\theta$, we have

$$\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \&c.$$

Cor. 1. Hence, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\theta - \frac{\theta^3}{1.2.3} + \&c.}{1 - \frac{\theta^2}{1.2} + \&c.}$,

$$= \theta + C.\theta^3 + C'\theta^5 + \&c. \qquad (M).$$

where the values of C, C', &c. may be determined by actual division.

Cor. 2. Hence, if (z) be an arc, whose radius is (a) and (y) and (x), its sine and cosine to that radius respectively, we have, since

$$\sin\theta = \frac{y}{a},\ \cos\theta = \frac{x}{a},\ \text{and } \theta = \frac{z}{a},$$

$$\frac{y}{a} = \frac{z}{a} - \frac{1}{1.2.3}.\frac{z^3}{a^3} + \frac{1}{1.2.3.4.5}.\frac{z^5}{a^5} - \&c.$$

and, therefore,

$$y = z - \frac{1}{1.2.3}.\frac{z^3}{a^3} + \frac{1}{1.2.3.4.5}.\frac{z^5}{a^4} - \&c.$$

And, in the same manner,

$$x = a - \frac{1}{1.2}.\frac{z^2}{a} + \frac{1}{1.2.3.4}.\frac{z^4}{a^3} - \&c.;$$

and, consequently, if (t) = tan z to radius (a), we have

$$t = z - C.\frac{z^3}{a^2} + C'.\frac{z^5}{a^4} - \&c.$$

where C and C' are the same as before.

Cor. 3. (Prop. 32. Cor. 1.), we have

$$\epsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

for (x) write $\theta.\sqrt{-1}$, then

$$\epsilon^{\theta\sqrt{-1}} = 1 + \theta\sqrt{-1} - \frac{\theta^2}{1.2} - \frac{\theta^3\sqrt{-1}}{1.2.3} + \&c.$$

next for (x) write $-\theta\sqrt{-1}$, then

$$\epsilon^{-\theta\sqrt{-1}} = 1 - \theta\sqrt{-1} - \frac{\theta^2}{1.2} + \frac{\theta^3\sqrt{-1}}{1.2.3} + \&c.;$$

hence, $\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}}$

$$= 2\left(1 - \frac{\theta^2}{1.2} + \&c.\right) = 2 . \cos\theta,$$

and $\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}$

$$= 2\sqrt{-1}\left(\theta - \frac{\theta^3}{1.2.3} + \&c.\right) = 2\sqrt{-1}\sin\theta;$$

hence, $\cos\theta = \frac{1}{2}(\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}})$,

and $\sin\theta = \frac{1}{2\sqrt{-1}}(\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}})$,

which expressions are only imaginary.

Cor. 4. If for (θ) we write $m\theta$, we have

$$\cos m\theta = \frac{1}{2} . (\epsilon^{m\theta\sqrt{-1}} + \epsilon^{-m\theta\sqrt{-1}}),$$

and $\sin m\theta = \frac{1}{2\sqrt{-1}}(\epsilon^{m\theta\sqrt{-1}} - \epsilon^{-m\theta\sqrt{-1}})$.

Cor. 5. If we make $\epsilon^{\theta\sqrt{-1}} = x$, we have

$$\epsilon^{m\theta\sqrt{-1}} = x^m, \quad \epsilon^{-\theta\sqrt{-1}} = \frac{1}{x},$$

and $\epsilon^{-m\theta\sqrt{-1}} = \frac{1}{x^m}$.

Hence, $2\cos\theta = x + \frac{1}{x}$, and $2\cos m\theta = x^m + \frac{1}{x^m}$,

and $2\sqrt{-1}\sin\theta = x - \frac{1}{x}$,

$$\text{and } 2\sqrt{-1}\sin m\theta = x^m - \frac{1}{x^m}.$$

Cor. 6. Hence,

$$\cos\theta + \sqrt{-1}\sin\theta = x,$$

$$\text{and } \cos\theta - \sqrt{-1}\sin\theta = \frac{1}{x},$$

$$\text{and } \cos m\theta + \sqrt{-1}\sin m\theta = x^m,$$

$$\text{and } \cos m\theta - \sqrt{-1}\sin m\theta = \frac{1}{x^m}.$$

Consequently, $(\cos\theta \pm \sqrt{-1}\sin\theta)^m$

$$= \cos m\theta \pm \sqrt{-1} \,.\, \sin m\theta.$$

Prop. XXXVII.

To find the value of an arc in terms of its sine.

Let $y = \sin\theta$, then $\theta = \sin^{-1} y$ *; and since the differential coefficient of (y) considered as a function of $(\theta) = \cos\theta = \sqrt{1-y^2}$, the differential coefficient of (θ) considered as a function of $y = \frac{1}{\sqrt{1-y^2}}$, or

$$\frac{d.\sin^{-1}y}{dy} = \frac{1}{\sqrt{1-y^2}} = (1-y^2)^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}y^2 + \frac{1.3}{2.4}y^4 + \frac{1.3.5}{2.4.6}.y^6 + \&\text{c.}$$

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* The symbols $\sin^{-1}y$, $\cos^{-1}y$, $\tan^{-1}y$, &c. are used to represent the length of a circular arc, whose sine, cosine, tangent, &c. respectively $=y$ to radius unity.

Hence, differentiating, we have

$$\frac{d^2 \sin^{-1}y}{dy^2} = y + \frac{1.3}{2}.y^3 + \frac{1.3.5}{2.4}.y^5 + \&c.$$

and $$\frac{d^3 \sin^{-1}y}{dy^3} = 1 + \frac{1.3^2}{2}y^2 + \frac{1.3.5^2}{2.4}y^4 + \&c.$$

And thus, by successive differentiations, we may find the other coefficients. Now, suppose (y) to increase by (k), then, by Taylor's theorem,

$$\sin^{-1}(y+k) = \sin^{-1}y + \frac{d.\sin^{-1}y}{dy}.k + \frac{d^2.\sin^{-1}y}{dy^2}.\frac{k^2}{1.2}$$

$$+ \frac{d^3.\sin^{-1}y}{dy^3}.\frac{k^3}{1.2.3} + \&c.$$

$$= \sin^{-1}y + k\left\{1 + \frac{y^2}{2} + \frac{1.3}{2.4}.y^4 + \&c.\right\}$$

$$+ \frac{k^2}{1.2}\left\{y + \frac{1.3}{2}y^3 + \&c.\right\}$$

$$+ \frac{k^3}{1.2.3}\left\{1 + \frac{1.3^2}{2}.y^2 + \&c.\right\}$$

$$+ \&c.$$

Now, let $y = 0$, then, since $\sin^{-1}y = 0$, we have

$$\sin^{-1}k = k + \frac{1}{1.2}.\frac{k^3}{3} + \frac{1.3}{2.4}.\frac{k^5}{5} + \&c.$$

Hence, writing (y) for (k), we have, since $y = \sin\theta$,

$$\theta = \sin\theta + \frac{1}{1.2}.\frac{\sin^3\theta}{3} + \frac{1.3}{2.4}.\frac{\sin^5\theta}{5} + \&c. \qquad (A).$$

Cor. 1. If (θ) becomes a quadrantal arc, $\sin\theta = 1$, and the value of (θ) is then represented by the series

$$1 + \frac{1}{1.2.3} + \frac{1.3}{1.2.4.5} + \frac{1.3.5}{1.2.4.6.7} + \&c.$$

The double of this series will express the value of the semi-circumference of a circle, whose radius $= 1$, and is represented by the Greek letter (π).

Cor. 2. If (z) be the circular arc, which subtends the angle θ to radius (a), and if $y = \sin z$ to this radius, then since

$$\theta = \frac{z}{a}, \text{ and } \sin\theta = \frac{y}{a},$$

we have, by series (A),

$$\frac{z}{a} = \frac{y}{a} + \frac{1}{1.2}.\frac{y^3}{3a^3} + \frac{1.3}{2.4}.\frac{y^5}{5a^5} + \&c.;$$

$$\therefore\ z = y + \frac{1}{1.2}.\frac{y^3}{3a^2} + \frac{1.3}{2.4}.\frac{y^5}{5a^4} + \&c. \qquad (B).$$

Cor. 3. If (z) becomes a quadrantal arc, $y = a$, and the value of (z) is represented by the series

$$a\left(1 + \frac{1}{1.2.3} + \frac{1.3}{2.4.5} + \&c.\right),$$

which $= \frac{\pi a}{2}$. Hence, four times this quadrantal arc, or the circumference of a circle, whose radius $= a$, is equal to $2\pi a$.

Cor. 4. Hence, the curved surface of a cylinder, whose length is (c), and radius $= a$, is $2\pi ac$, (Prop. 7.).

Prop. XXXVIII.

To find the value of an arc in terms of its tangent.

Let $(t) = \tan\theta$, then $\theta = \tan^{-1}t$, and since the differential coefficient of (t) considered as a function of $(\theta) = \sec^2\theta$, (Prop. 35. Cor. 5.), the differential coefficient of (θ), considered as a function of (t) $= \dfrac{1}{\sec^2\theta}$,

$$\text{or } \frac{d.\tan^{-1}t}{dt} = \frac{1}{\sec^2\theta} = \frac{1}{1+t^2} = (1+t^2)^{-1}$$

$$= 1 - t^2 + t^4 - t^6 + \&c.$$

Hence, differentiating successively, we have

$$\frac{d^2\tan^{-1}t}{dt^2} = -2t + 4t^3 - 6t^5 + \&c.$$

$$\frac{d^3\tan^{-1}t}{dt^3} = -2 + 3.4.t^2 - 5.6.t^4 + \&c.$$

And thus, by continuing to differentiate, we may find the remaining differential coefficients. Now, suppose (t) to increase by (k), then, by Taylor's theorem,

$$\tan^{-1}(t+k) = \tan^{-1}t$$

$$+ \frac{d.\tan^{-1}.t}{dt}.k + \frac{d^2.\tan^{-1}t}{dt^2}.\frac{k^2}{1.2}$$

$$+ \frac{d^3.\tan^{-1}t}{dt^3}.\frac{k^3}{1.2.3} + \&c.$$

$$= \tan^{-1}t + k.(1 - t^2 + t^4 - \&c.).$$

$$+\frac{k^2}{1\,.\,2}\,.\,(-2t+4t^3-\&c.)$$

$$+\frac{k^3}{1\,.\,2\,.\,3}\,.\,(-2+3\,.\,4\,.\,t^2-\&c.)$$

$$+\ \&c.$$

Now, let $t=0$, then $\tan^{-1}t=0$; and, therefore, we have

$$\tan^{-1}k=k-\frac{k^3}{3}+\frac{k^5}{5}-\&c.$$

and writing (t) for (k), since $\tan^{-1}t=\theta$, and $t=\tan\theta$, we have

$$\theta=\tan\theta-\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}-\frac{\tan^7\theta}{7}+\&c.\qquad (A).$$

Cor. 1. If we suppose (z) to be the arc, which subtends the angle (θ) to radius (a), and (t) to be its tangent to that radius, then, since

$$\theta=\frac{z}{a},\ \text{ and }\ \tan\theta=\frac{t}{a},$$

we have, from series (A),

$$\frac{z}{a}=\frac{t}{a}-\frac{1}{3}\cdot\frac{t^3}{a^3}+\frac{1}{5}\cdot\frac{t^5}{a^5}-\frac{1}{7}\cdot\frac{t^7}{a^7}+\&c.$$

$$\therefore\ z=t-\frac{1}{3}\cdot\frac{t^3}{a^2}+\frac{1}{5}\cdot\frac{t^5}{a^4}-\frac{1}{7}\cdot\frac{t^7}{a^6}+\&c.$$

Cor. 2. If we make $t=a$, (z) becomes the arc which subtends an angle of 45 degrees, and is, therefore, equal to one fourth part of the semicircumference of a circle, whose radius $=a$, and hence it $=\frac{\pi a}{4}$; hence,

$\frac{na^2}{2} . \sin \frac{2\pi}{n}$, and be, therefore, expressed by the series

$$\frac{na^2}{2}\left\{\frac{2\pi}{n} - \left(\frac{2\pi}{n}\right)^3 . \frac{1}{1.2.3} + \&c.\right\}, \text{ (by Prop. 36.)}$$

$$\text{or } \pi a^2 - \frac{4\pi^3}{n^2} . \frac{a^2}{1.2.3} + \&c. \qquad (B).$$

Now, the area of the circle is less than one of these polygons, and greater than the other; and, consequently, the number expressing it, lies between the series (A) and the series (B), whatever be the value of (n), which cannot be the case, unless the area of the circle $= \pi a^2$.

COR. 1. Hence, the area of a circle = its radius, × its semi-circumference.

COR. 2. Hence, the content of a cylinder, whose length $= c$, and whose rad. $= a$, is $\pi c a^2$.

COR. 3. Hence, in a circle, whose radius $= a$, the area of any sector, containing an angle $= \theta$, is $\frac{1}{2} a^2 \theta$. For the area of the whole circle $= \pi a^2$, and this area : area of sector, whose angle is (θ) :: $2\pi : \theta$; and, therefore, the area of the sector is equal to $\frac{\theta}{2\pi}$. × area of the circle $= \frac{\theta}{2\pi} . \pi a^2 = \frac{a^2\theta}{2}$.

COR. 4. Hence, if the radius of the circle had been unity, the area of a sector, whose angle is θ, would have equalled $\frac{1}{2}\theta$.

COR. 5. Hence, the whole surface of a cylinder, whose length and radius are (c) and (a) respectively, is

$$2\pi ac + 2\pi a^2, \text{ or } 2\pi a . (a + c).$$

Cor. 6. Hence, we may find the proportion between the length and the radius of a cylinder, which, with a given surface, shall contain the greatest possible content.

Let (x) be its length, (y) its radius, and $2\pi a^2$ the given surface, then

$$2\pi y\,.\,(x+y)=2\pi a^2\,;$$

$$\therefore\ x=\frac{a^2}{y}-y.$$

But the content of this cylinder $=\pi y^2 x=\pi\,.\,(a^2 y-y^3)$. Hence, the differential coefficient of this content, when considered as a function of (y), must equal nothing; and, therefore,

$$a^2-3y^2=0, \text{ and } \therefore y=\pm\frac{a}{\sqrt{3}}\,;$$

$$\text{hence, } x=\pm\left(a\sqrt{3}-\frac{a}{\sqrt{3}}\right)=\pm\frac{2a}{\sqrt{3}}\,;$$

$$\text{hence, } y : x :: 1 : 2, \text{ or } x=2y.$$

Hence, the length of the cylinder must equal its diameter.

The equation $y=\frac{a}{\sqrt{3}}$, gives a maximum value of the content, because, if we again differentiate, we find the second differential coefficient of the content to be negative.

From the preceding propositions, we may find the differentials of all circular functions.

Ex. 1. $d.\sin^n x=n\,.\,\sin^{n-1}x\,.\,d\sin x$

$$=n\,.\,\sin^{n-1}x\,.\,\cos x\,.\,dx.$$

Ex. 2. $d\,.\cos^n x = n\,.\cos^{n-1} x\,.\,d\,.\cos x$

$$= -n\,.\cos^{n-1} x\,.\sin x\,.\,dx.$$

Ex. 3. $d\,.\tan^n x = n\,.\tan^{n-1} x\,.\,d\,.\tan x$

$$= n\,.\tan^{n-1} x\,.\sec^2 x\,.\,dx.$$

If we represent an arc, whose sine $= x$, by the symbol $\sin^{-1} x$, we have

$$d\,.\sin^{-1} x = \frac{dx}{\sqrt{1-x^n}}.$$

Again, if $\cos^{-1} x$ represent an arc, whose cosine $= x$,

$$d\,.\cos^{-1} x = \frac{-dx}{\sqrt{1-x^2}}.$$

Similarly, $d\,.\tan^{-1} x = \dfrac{d\,.x}{1+x^2}.$

By combining the propositions of this section with those of the preceding, we may differentiate quantities involving both logarithmic and circular functions.

Ex. 1. $d\,.\text{hyp. log.} \sin x = \dfrac{d\,.\sin x}{\sin x}$

$$= \frac{\cos x\,.\,dx}{\sin x} = \text{cotan}\, x\,.\,dx.$$

Ex. 2. $d\,.\text{hyp. log.} \cos x = \dfrac{d\,.\cos x}{\cos x}$

$$= \frac{-\sin x dx}{\cos x} = -\tan x dx.$$

Ex. 3. $d\,.e^{\sin x} = e^{\sin x} d\,.\sin x = e^{\sin x}.\cos x\,.\,dx.$

Ex. 4. $d.(\cos x)^{\sin x} = dx.(\cos x)^{\sin x}.$

$$\left(\cos x \,.\, \text{hyp. log. } \cos x - \frac{\sin^2 x}{\cos x}\right).$$

For, if we assume $u = (\cos x)^{\sin x}$,

we have hyp. log. $u =$ hyp. log. $(\cos.\, x)^{\sin x}$

$$= \sin x \,.\, \text{hyp. log. } \cos x;$$

$$\therefore \frac{du}{u} = \cos x \,.\, \text{hyp. log. } \cos x \,.\, dx - \tan x \,.\, dx \,.\, \sin x$$

$$= dx\left(\cos x \,.\, \text{hyp. log. } \cos x - \frac{\sin^2 x}{\cos x}\right);$$

$$\therefore du = dx \,.\, (\cos x)^{\sin x}.\left(\cos x \,.\, \text{h. l. } \cos x - \frac{\sin^2 x}{\cos x}\right).$$

From the preceding propositions, we may determine the value of many vanishing fractions, which involve circular functions.

Ex. 1. $\dfrac{\sin 2\theta - \sin \theta}{\sin 3\theta - \sin \theta}$, when $\theta = 0$.

We have $\dfrac{d.(\sin 2\theta - \sin \theta)}{d\theta} = 2 \cos 2\theta - \cos \theta$

$$= 1, \text{ (when } \theta = 0).$$

And $\dfrac{d.(\sin 3\theta - \sin \theta)}{d\theta} = 3.\cos 3\theta - \cos \theta$

$$= 2, \text{ (when } \theta = 0);$$

hence, $\dfrac{\sin 2\theta - \sin \theta}{\sin 3\theta - \sin \theta} = \dfrac{1}{2}$, (when $\theta = 0$).

Ex. 2. $\dfrac{\cos \theta - \cos 2\theta}{\cos \theta - \cos 3\theta}$, when $\theta = 0$.

We have $\dfrac{d.(\cos\theta-\cos 2\theta)}{d\theta} = -\sin\theta+2\sin 2\theta,$

and $\dfrac{d^2.(\cos\theta-\cos 2\theta)}{d\theta^2} = -\cos\theta+4.\cos 2\theta$

$=3,$ (when $\theta=0$).

Again, $\dfrac{d.(\cos\theta-\cos 3\theta)}{d\theta} = -\sin\theta+3\sin 3\theta,$

and $\dfrac{d^2.(\cos\theta-\cos 3\theta)}{d\theta^2} = -\cos\theta+9\sin 3\theta$

$=8,$ (when $\theta=0$);

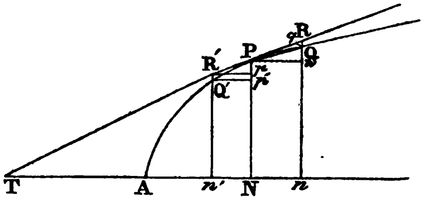
hence, $\dfrac{\cos\theta-\cos 2\theta}{\cos\theta-\cos 3\theta} = \dfrac{3}{8},$ (when $\theta=0$).

SECT. V.

ON REGULAR CURVES.

A REGULAR curve is one, whose co-ordinates are functions of each other. Those curves, which are described *liberâ manu*, and in which, if referred to rectangular axes, there subsists no relation between their co-ordinates, manifestly cannot be the subject of mathematical reasoning.

Let APQ be a regular curve, ANn the horizontal axis, A the first point from which the abscissa



$AN=x$ is measured, and through which we may suppose the perpendicular axis to be drawn; let $Q'n'$, PN, Qn be three equidistant ordinates; and let $PN=y$, $Nn=h=Nn'$. Then since the ordinate to any point is the same function of its abscissa throughout, Qn, $Q'n'$ are the same functions of their abscissæ An, An' respectively, that PN is of AN; if, therefore, we represent (y) by the symbol $\phi\,(x)$ we have

$$Qn = \phi(x+h), \ \{\text{because } An = (x+h)\},$$

$$= \phi x + f\phi(x) \,.\, h + f^2 .\, \phi(x) \frac{h^2}{1\,.\,2} + \&c.$$

and $Q'n' = \phi(x-h)$, because $An' = (x-h)$

$$= \phi(x) - f.\,\phi(x) h + f^2 .\, \phi(x) \frac{h^2}{1\,.\,2} - \&c.$$

Here $f\phi(x), f^2.\,\phi(x)$, &c. represent the differential coefficients of the ordinate, considered as a function of the abscissa, at the point P, and, therefore,

$$f.\,\phi(x+h), \quad f^2.\,\phi(x+h), \ \&c.$$

will represent them at the point Q, and

$$f\phi(x-h), \quad f^2\phi(x-h), \ \&c.$$

at the point Q'.

The area of a curve corresponding to any abscissa, is a function of that abscissa; and, consequently, if we represent the area APN, which corresponds to the abscissa $AN = x$, by $\psi(x)$, the areas $AQ'n'$, AQn, which correspond to the abscissæ, An', An will be represented respectively by $\psi(x-h)$, and $\psi(x+h)$.

The length of a curve is also a function of its abscissa, and if we represent the length of the curve AP by $\phi'(x)$, the curves AQ and AQ' will equal $\phi'(x+h)$ and $\phi'(x-h)$ respectively. The same is also true of the solid and surface of revolution generated by the curve APQ revolving round its abscissa ANn.

In this section, whenever we speak of the ordinates, lengths, areas, &c. of curves, we shall always regard them as functions of their corresponding abscissæ, and, whenever we speak of the differential coefficients of these quantities, we mean their differential coefficients, when taken upon this supposition.

In the following propositions we shall take for granted these axioms.

Axiom 1. If a tangent be drawn at one extremity of an arc, and from the other extremity a subtense be drawn making a finite angle with the tangent, the length of the arc is less than the tangent together with the subtense.

Thus, in the preceding figure, the arc PQ is less than $PR+RQ$.

Axiom 2. The length of any arc is greater than its chord.

Axiom 3. If a curve APQ, (see preceding figure,) revolves round its abscissa ANn, the surface traced out by the arc PQ is less than the curved surface of a cylinder, whose length equals the length of the arc, and whose radius $= Qn$, the greatest ordinate, and greater than the curved surface of the cylinder, whose length equals the length of the arc, and whose radius $= PN$, the least ordinate.

Prop. XL.

The equation to any straight line may be reduced to this form

$$y = ax + b,$$

where (x) *is the abscissa measured along the horizontal axis,* (y) *the ordinate measured along the perpendicular axis, and* (a) *the tangent of the inclination of the given straight line to the horizontal axis.*

Through the intersection of the axes draw another straight line parallel to the given line, and, therefore, equally inclined to the horizontal axis; and let (y') be the ordinate to this straight line, corresponding to the abscissa (x); then it is manifest, that

$$y' : x :: a : 1.$$

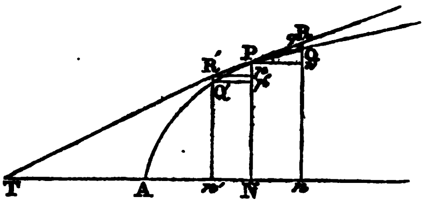
Hence, the equation to this straight line is $y' = ax$; but it may easily be proved, that the difference between (y) and (y') is a constant quantity; let then this difference $= b$, then $y - y' = b$, and $y' = y - b$;

$$\text{hence, } y - b = ax, \text{ and } \therefore y = ax + b.$$

Def. A tangent to a curve is a straight line, which meets the curve at any point, but, being produced, does not cut it at that point.

Def. A subtangent to any point in a curve, is that part of the horizontal axis, which is intercepted between the ordinate to that point, and the intersection of the tangent to that point, with the horizontal axis.

Thus, let *AP* be a curve, *ANn* the horizontal axis, *A* the first point, from whence the co-ordinates are measured, and through which we may suppose the other axis to be drawn.



Let *P* be any point in the curve, draw *PN* perpendicular to *AN*, and suppose the straight line *TPR* to meet, but not to cut, the curve at *P*, and to intersect *AN* in *T*, then *TPR* is a tangent to the curve, and *NT* is a subtangent.

PROP. XLI.

If (θ) *be the angle, at which the tangent to any point of a curve is inclined to the horizontal axis, the differential coefficient of the ordinate at that point is equal to* $\tan\theta$.

Let APQ be the curve, $Q'n'$, PN, Qn three equidistant ordinates, TPR a tangent at P, cutting the ordinates $Q'n'$, Qn produced in R' and R. Complete the rectangles $R'N$, Pn. Let Nn or $Nn' = h$, $AN = x$, and $PN = y = \phi(x)$.

$$\text{Then } Qn = \phi(x) + f.\phi(x).h + f^2.\phi(x).\frac{h^2}{1.2} + \&c.$$

$$\text{and } Q'n' = \phi(x) - f.\phi(x).h + f^2.\phi(x).\frac{h^2}{1.2} - \&c.$$

Also since $\angle RP\pi = \angle PR'p = \theta$, we have

$$Rn = PN + R\pi = \phi(x) + \tan\theta.h,$$

$$\text{and } R'n' = PN - Pp = \phi(x) - \tan\theta.h.$$

Hence, we have

$$Rn - Qn = \{\tan\theta - f.\phi(x)\}.h - f^2.\phi(x).\frac{h^2}{1.2} - \&c. \quad (A),$$

and

$$R'n' - Q'n' = \{f.\phi(x) - \tan\theta\}.h - f^2.\phi(x).\frac{h^2}{1.2} + \&c. \quad (B).$$

Now suppose, as in the present figure, the arcs PQ, PQ', to lie both beneath the tangent TPR; then Rn is greater than Qn; hence, the series (A) is positive; and, consequently, (by Prop. 13. Cor. 3.) $f.\phi(x)$ cannot be greater than $\tan\theta$. Again, $R'n'$

is greater than $Q'n'$; hence, the series (B) is positive; and, therefore, $f.\phi(x)$ is not less than $\tan\theta$. Hence, we have

$$f.\phi(x) \text{ or } \frac{dy}{dx} = \tan\theta.$$

In the same way it might have been proved, if the curve had lain above the tangent.

COR. 1. Hence, if the tangent to any point in a curve, whose co-ordinates are (x) and (y) be drawn, the equation to this tangent is

$$y' - y = \frac{dy}{dx} . (x' - x),$$

where (y), (x), and $\frac{dy}{dx}$ are quantities, which remain the same for the same tangent, and (y') and (x') the variable co-ordinates of this tangent; for, (by Prop. 40.)

$$y' = \tan\theta . x' + b;$$

but, when $y' = y$, $x' = x$; hence, in that case, we have

$$y = \tan\theta . x + b,$$

and eliminating (b) by subtraction, we have

$$y' - y = \tan\theta . (x' - x)$$

$$= \frac{dy}{dx} . (x' - x).$$

COR. 2. Hence, we may find the point, where the tangent cuts the axis. At this point $y' = 0$, and $\therefore x' = x - \frac{ydx}{dy}$. Hence, the distance of the point

T from A measured in the direction $AN = x - \frac{ydx}{dy}$, and, consequently, if $\frac{ydx}{dy}$ be less than x, T lies to the right of A; but, if $\frac{ydx}{dy}$ be greater than (x), T lies to the left of A.

Cor. 3. Hence, subtangent $= \frac{-ydx}{dy}$.

Cor. 4. If the curve admits a tangent at any particular point, the position of this tangent may be found by the preceding expressions, but the converse is not necessarily true. The position of a straight line may be found by the application of the preceding expressions, and yet that straight line not be a tangent.

Cor. 5. When the ordinate is a maximum or a minimum, the tangent to the curve is parallel to the horizontal axis. In the former case, the curve lies below the tangent, in the latter, above. For, when $\frac{dy}{dx} = 0$, $\tan\theta = 0$; and, consequently, the tangent to the curve at this point cannot cut the horizontal axis, for if it did, it must cut it at some finite angle, whose tangent is not evanescent.

Cor. 6. Hence, a tangent to a curve may be drawn by the following construction. Through the point P, draw the straight line $P\pi$ parallel to the axis AN; take $P\pi =$ a linear unit, and from the point π erect the perpendicular πR, and make $\pi R = f.\phi(x)$ or $\frac{dy}{dx}$, join RP, and produce it both ways; then, if the curve admits a tangent, RP is that tangent. If $\frac{dy}{dx}$ be positive, πR is to be drawn upwards, but if $\frac{dy}{dx}$ be negative, it is to be drawn downwards.

Cor. 7. A curve cannot have more than one tangent at a point. For it has been shown, that if RP be a tangent at P, the tangent of the angle $RP\pi$ must $= \frac{dy}{dx}$; and, consequently, (unless $\frac{dy}{dx}$ has more than one value, while (x) and (y) are the same,) the curve cannot have more than one tangent.

Cor. 8. If two curves touch each other at any point, they have the same tangent; and if they are referred to the same axes, $\frac{dy}{dx}$ is the same in both. For it is manifest, that the straight line, which touches the outer curve, cannot cut the inner; and, since $\frac{dy}{dx}$ determines the inclination of this tangent to the same axis in each, it must be the same for both.

Ex. 1. To draw a tangent to an ellipse. Suppose the axis-major of this ellipse to be the horizontal axis, and $=2a$, and the axis-minor to be the perpendicular axis, and $=2b$; then we have

$$y = \pm \frac{b}{a} \cdot \sqrt{a^2 - x^2},$$

$$\therefore \frac{dy}{dx} = \mp \frac{b}{a} \cdot \frac{x}{\sqrt{a^2 - x^2}};$$

$$\therefore \frac{y\,dx}{dy} = - \frac{a^2 - x^2}{x}.$$

Now, if (x) be measured to the right of the minor axis, the distance of the point T, where the tangent intersects the axis-major produced, from the centre of the ellipse, will equal

$$+x - \frac{y\,dx}{dy}$$

$$= +x + \frac{a^2 - x^2}{x} = +\frac{a^2}{x}.$$

Hence, *AT* : semi-axis major :: semi-axis major : abscissa, as appears also from conic sections. The positive sign shows the direction, in which *AT* is to be measured.

Def. A curve is said to be concave at any point, if the tangent at that point lies above the curve.

Def. A curve is said to be convex at any point, if the tangent at that point lies below the curve.

Def. If, when an ordinate is drawn to any point in a curve, the curve on one side of this ordinate is concave, and on the other convex; this point is called a point of contrary flexure.

In general, we term curves concave or convex, according as they turn their convex sides upwards or downwards. Thus, if a diameter be drawn in a circle parallel to the horizontal axis, that part of the circle, which lies above the diameter, is termed concave, because it turns its convexity upwards, and that part, which lies below the diameter, is termed convex, because it turns its convexity downwards. Hence, it is not always true, that, when one part of a curve is convex, and another concave, a point of contrary flexure intervenes, for no such point exists in a circle. By the definition, in order to constitute a point of contrary flexure, it is necessary, that the ordinate to this point, should have a portion of the curve, lying on each side of it; and that the curve on one side should be convex, and on the other concave. But in the circle, it is manifest, that the points, where the change from convexity to concavity takes place, are the extremities of the diameter, drawn parallel to the horizontal axis; and that the ordinates to these points have both the concave and the convex arcs lying on the same side.

PROP. XLII.

In any curve, the second differential coefficient of the ordinate is positive or negative, according as the curve is convex or concave.

Let the curve be concave, and lie beneath the tangent, then (using the same figure and construction as before), we have

$$Rn = \phi(x) + \tan\theta . h = \phi(x) + f . \phi(x)\, h,$$

$$\text{and } Qn = \phi(x) + f . \phi(x) . h + f^2 . \phi(x)\, \frac{h^2}{1.2} + \&c.$$

$$\therefore Rn - Qn = -f^2 . \phi(x) \frac{h^2}{1.2} - f^3 . \phi(x) . \frac{h^3}{1.2.3} - \&c.$$

the sum of which series must be positive; hence, the coefficient of the first term must be positive; that is, $-f^2 . \phi(x)$ must equal some positive quantity; and, therefore, $f^2 . \phi(x)$ must be negative.

In the same manner it might have been shown, that if the curve had lain above the tangent, $f^2 . \phi(x)$ would have been positive.

COR. 1. Hence, conversely, if $f^2 . \phi(x)$ be positive, the curve is convex, and if $f^2 \phi(x)$ be negative, the curve is concave. For if $f^2 \phi(x)$ be negative, and $= -X$, we have

$$Rn - Qn = X . \frac{h^2}{2} - f^3 . \phi(x) \frac{h^3}{1 . 2 . 3} - \&c.$$

which is positive, when (h) is sufficiently diminished; hence, Rn is greater than Qn, and, therefore, the curve lies below the tangent. In the same manner it may be shown, that if $f^2\phi(x)$ be positive, the curve lies above the tangent.

Ex. 1. Suppose the curve to be an ellipse, then we have

$$y = \pm \frac{b}{a} . \sqrt{2ax - x^2}.$$

Hence, taking the positive value of (y),

$$\frac{dy}{dx} = \frac{b}{a} . \frac{(a-x)}{\sqrt{2ax - x^2}};$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-ab}{(2ax - x^2)^{\frac{3}{2}}},$$

which is always negative; hence, the part of the curve above the axis major is always concave; if the negative value of (y) had been taken, $\frac{d^2y}{dx^2}$ would have been positive; and, therefore, that part of the ellipse, which lies beneath the axis major is convex.

Ex. 2. Suppose the curve to be a parabola.

Here $y^2 = 4ax$; hence, $y = \pm 2\sqrt{ax}$.

Taking the positive value of (y),

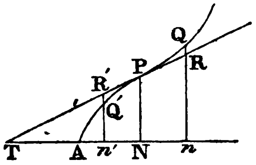
$$\frac{dy}{dx} = \sqrt{\frac{a}{x}}, \text{ and } \frac{d^2y}{dx^2} = - \frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}.$$

Hence, the curve above the axis is concave, and, as before, it may be shown to be convex beneath the axis.

PROP. XLIII.

At a point of contrary flexure, the second differential coefficient of the ordinate vanishes.

Let P be a point of contrary flexure Qn, $Q'n'$ two ordinates equidistant from PN, the ordinate to the point P. Supposing Nn, Nn' each equal h, and $AN=x$, we have



$$An=x+h, \text{ and } An'=x-h;$$

and, therefore, the second differential coefficients of the ordinate at the points P, Q, and Q' will be represented respectively by

$$f^2\phi(x), \quad f^2\phi(x+h) \text{ and } f^2\phi(x-h). \quad \text{But}$$

$$f^2.\phi(x+h)=f^2.\phi(x)+f^3.\phi(x)\frac{h}{1}$$

$$+f^4.\phi(x).\frac{h^2}{1.2}+\&c.$$

$$\text{and } f^2.\phi(x-h)=f^2.\phi(x)-f^3.\phi(x)\frac{h}{1}$$

$$+f^4.\phi(x)\frac{h^2}{1.2}-\&c.$$

Hence, when $f^2\phi(x)$ does not vanish, but is positive, if we take (h) so small, that the first term in both these series is greater than the sum of all the rest, the values of $f^2.\phi(x+h)$ and $f^2.\phi(x-h)$ are both positive; and, therefore, the second differential coefficients to the points Q and Q' are both positive; and, therefore, the curve is convex on both sides of P, and, consequently, P is not a point of contrary flexure. In the same manner it might have been shown, that if $f^2\phi(x)$ had been finite and negative, the curve would have been concave on both sides of P. Hence, P cannot be a point of contrary flexure, unless $f^2\phi(x)=0$.

COR. 1. Conversely, if at any point in a curve $f^2\phi(x)$ or $\frac{d^2y}{dx^2}=0$, while $f^3\phi(x)$ or $\frac{d^3y}{dx^3}$ remains finite, this point is a point of contrary flexure. For, as before,

$$f^2.\phi(x+h)=f^2.\phi(x)+f^3.\phi(x)\frac{h}{1}$$

$$+f^4\phi(x)\frac{h^2}{1.2}+\&\text{c.}$$

$$=f^3\phi(x).\frac{h}{1}$$

$$+f^4.\phi(x)\frac{h^2}{1.2}+\&\text{c.}\quad\{\text{since } f^2\phi(x)=0\},$$

$$\text{and } f^2.\phi(x-h)=-f^3.\phi(x)\frac{h}{1}+f^4.\phi(x)\frac{h^2}{1.2}-\&\text{c.}$$

Hence, by making (h) sufficiently small, we shall have one of the quantities

$$f^2\phi(x+h),\ \text{and } f^2.\phi(x-h)$$

positive, and the other negative; and, consequently, the curve on one side of this point is convex, and on the other side concave; hence, this point is a point of contrary flexure.

Cor. 2. If at any point, both

$$f^2.\phi(x)=0, \quad \text{and } f^3\phi(x)=0,$$

while $f^4.\phi(x)$ remains finite, this point is not a point of contrary flexure. For, as before,

$$\{\text{since } f^2\phi(x)=0, \quad \text{and } f^3\phi(x)=0\},$$

$$f^2.\phi(x+h)=f^4.\phi(x)\frac{h^2}{1.2}+f^5.\phi(x)\frac{h^3}{1.2.3}+\&c.$$

$$\text{and} \quad f^2\phi(x-h)=f^4.\phi(x)\frac{h^2}{1.2}-f^5.\phi(x)\frac{h^3}{1.2.3}+\&c.$$

Now, diminishing (h), so that the first term is greater than the sum of all the rest, we have

$$f^2.\phi(x+h), \quad \text{and } f^2.\phi(x-h)$$

both positive, or both negative, according as $f^4\phi(x)$ is positive or negative; hence, if $f^4.\phi(x)$ be positive, the curve is convex on both sides of this point, but if $f^4\phi(x)$ be negative, the curve is concave; and hence, in neither case, is this point a point of contrary flexure.

Cor. 3. By a similar process of reasoning it may be shown, that if at any point in a curve the number of differential coefficients which vanish, beginning with the second, be odd, this point is a point of contrary flexure; but if the number be even, this point is not a point of contrary flexure.

Ex. 1. Let the equation to a curve be $y=\frac{x^3}{a^2}$,

$$\text{then } \frac{dy}{dx} \text{ or } f'.\phi(x)=\frac{3x^2}{a^2},$$

$$\text{and } \frac{d^2y}{dx^2} \text{ or } f^2.\,\phi(x) = \frac{6x}{a^2},$$

$$\text{and } \frac{d^3y}{dx^3} \text{ or } f^3.\,\phi(x) = \frac{6}{a^2},$$

$$\text{and } \frac{d^4y}{dx^4} \text{ or } f^4\phi(x) = 0.$$

Here, when $x=0$, we have $f^2\phi(x)=0$, while $f^3.\,\phi(x)$ remains positive; hence, the point, whose abscissa $=0$, is a point of contrary flexure.

Ex. 2. Let the equation to a curve be $y = \frac{x^4}{a^3}$,

$$\text{then } \frac{dy}{dx} = \frac{4x^3}{a^3}, \quad \frac{d^2y}{dx^2} = \frac{12.x^2}{a^3}, \quad \frac{d^3y}{dx^2} = \frac{24x}{a^3},$$

$$\text{and } \frac{d^4y}{dx^4} = \frac{24}{a^3}.$$

Hence, at the point where $x=0$, $f^2.\,\phi(x)$, and $f^3\phi(x)=0$, but $f^4.\,\phi(x)$ remains finite; hence, this point is not a point of contrary flexure.

If the equation to a curve had been $y = \frac{x^{2n+1}}{a^{2n}}$, we should have had

$$\frac{dy}{dx} = \frac{(2n+1)\,x^{2n}}{a^{2n}}, \quad \frac{d^2y}{dx^2} = \frac{2n.(2n+1)\,x^{2n-1}}{a^{2n}},$$

$$\frac{d^3y}{dx^3} = \frac{(2n-1)\;2n.(2n+1)\;x^{2n-2}}{a^{2n}},$$

$$\text{and at length } \frac{d^{2n+1}y}{dx^{2n+1}} = \frac{1\,.\,2\,.\,3\,.........(2n+1)}{a^{2n}}.$$

Hence, each of the differential coefficients,

$$f^2.\,\phi(x), \quad f^3\phi(x), \quad f^4.\,\phi(x).........f^{2n}.\,\phi(x),$$

becomes equal to nothing, when $x=0$, and as the number of these coefficients is odd, the point, whose abscissa $=0$, must be a point of contrary flexure. In the same manner it might have been shown, that if the equation to a curve had been $y=\frac{x^{2n}}{a^{2n-1}}$, there would have been no point of contrary flexure, when $x=0$.

COR. 4. It has been already observed, that there may be points, in which Taylor's theorem fails, owing to some of the differential coefficients becoming infinite. When this theorem is applied to express the relation between the corresponding increments of the co-ordinates of curves, the points, where it fails, are frequently points of contrary flexure. Take, for example, the curve whose equation is

$$y=(x-a)^{\frac{1}{3}},$$

$$\text{then } \frac{dy}{dx}=\frac{1}{3.(x-a)^{\frac{2}{3}}}, \quad \text{and } \frac{d^2y}{dx^2}=\frac{-2}{2.(x-a)^{\frac{5}{3}}}.$$

Here we may observe, that while (x) is less than (a), $f^2.\phi(x)$ is positive; and, therefore, the curve is convex; and, when (x) is greater than (a), $f^2\phi(x)$ is negative, and, therefore, the curve is concave; and, consequently, there must be a point of contrary flexure, when $x=a$. Here, however, $f^2.\phi(x)$ does not vanish, but becomes infinite. If, however, instead of considering (y) as a function of (x), we had considered (x) as a function of (y), and had differentiated accordingly, we should have had

$$x=y^3+a;$$

and hence,

$$\frac{dx}{dy} = 3y^2, \text{ and } \frac{d^2x}{dy^2} = 6y = 6\,.\,(x-a)^{\frac{1}{3}},$$

and $\dfrac{d^3x}{dy^3} = 6.$

Where, since none of the differential coefficients become infinite, when $x=a$, and $\therefore\ y=0$, Taylor's theorem will not fail, and, consequently, since, when $x=a$, $\dfrac{d^2x}{dy^2} = 0$, and $\dfrac{d^3x}{dy^3}$ remains finite, there will be a point of contrary flexure, when $x=a$.

Again, take for an example the curve, whose equation is

$$y=(x-a)^{\frac{5}{3}},$$

then $\dfrac{dy}{dx} = \dfrac{5}{3}\,.\,(x-a)^{\frac{2}{3}}$, and $\dfrac{d^2y}{dx^2} = \dfrac{10}{9}\,.\,\dfrac{1}{(x-a)^{\frac{1}{3}}}$.

Here, as in the former example, we may observe, that the curve is concave or convex, according as (x) is less or greater than (a); and, consequently, when $x=a$, there is a point of contrary flexure. But at this point $f^2.\phi(x)$ does not vanish, but becomes infinite; and, consequently, Taylor's theorem fails. If, however, we consider (x) as a function of (y), we have

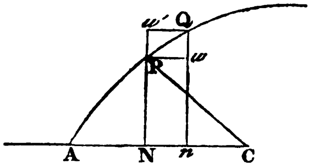
$$y^3=(x-a)^5, \text{ and } x=y^{\frac{3}{5}}+a; \text{ hence,}$$

$$\frac{dx}{dy} = \frac{3}{5y^{\frac{2}{5}}}, \text{ and } \frac{d^2x}{dy^2} = -\frac{6}{25y^{\frac{7}{5}}};$$

hence, when $x=a$, and $y=0$, these differential coefficients become both infinite; and, consequently, Taylor's theorem again fails. In such cases as this, in order to determine whether there is a point of contrary flexure, when $f^2 \phi(x)$ becomes infinite, we must observe whether this differential coefficient changes its sign, as it passes through infinity.

Prop. XLIV.

The area of a curve, corresponding to any abscissa, has its differential coefficient equal to the ordinate.



Let APQ be a curve, ANn its abscissa, PN, Qn two ordinates; complete the rectangles Pn, QN; let $AN = x$, $PN = y$, and $Nn = h$, and suppose the ordinate PN, and the area APN to be represented by the symbols $\phi(x)$ and $\psi(x)$ respectively.

Then the area $QPNn = \psi(x+h) - \psi(x)$

$$= f . \psi(x) . h + f^2 . \psi(x) \frac{h^2}{1 . 2} + \&c. \qquad (A).$$

The rectangle $Pn = PN . Nn = \phi(x) . h.$

The rectangle $QN = Qn . nN = \phi(x+h) . h$

$$= \phi(x) . h + f . \phi(x) . h^2 + \&c. \qquad (B),$$

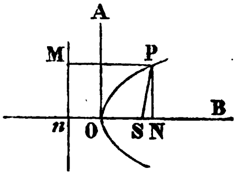
but the area $QPNn$ lies between the rectangles Pn and QN; hence, the series (A) lies between the quantity $\phi(x)h$, and the series (B); and, therefore, (by Prop. 13. Cor. 2.), we have

$$f.\psi(x)=\phi(x)=y.$$

Cor. 1. Hence, $\dfrac{d.\psi(x)}{dx}=y$,

$$\text{and } \therefore d.\psi(x)=y\,dx.$$

Ex. 1. In the common parabola, we have



$$y^2=4ax, \quad \therefore y=2a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$\therefore y\,dx=2a^{\frac{1}{2}}x^{\frac{1}{2}}dx,$$

$$\text{or } d.\psi(x)=2a^{\frac{1}{2}}x^{\frac{1}{2}}dx.$$

If we were to differentiate the quantity $\dfrac{4a^{\frac{1}{2}}x^{\frac{3}{2}}}{3}$, we should find

$$d.\frac{4a^{\frac{1}{2}}x^{\frac{3}{2}}}{3}=\frac{4a^{\frac{1}{2}}}{3}.dx^{\frac{3}{2}}=2a^{\frac{1}{2}}x^{\frac{1}{2}}dx.$$

Here we may observe, that both $\psi(x)$ and $\dfrac{4a^{\frac{1}{2}}x^{\frac{3}{2}}}{3}$ have the same differential coefficient; and, consequently, (by Prop. 24.), their difference is constant,

$$\text{and } \therefore \psi(x)=\frac{4a^{\frac{1}{2}}x^{\frac{3}{2}}}{3}+C, \qquad (A),$$

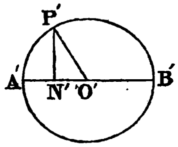
where C is constant. The value of C must be determined from the nature of the problem. In the present example, let $\psi(x)$ represent the area OPN, where $ON=x$, and $PN=y$; then, since the equation (A) holds good, whatever be the value of (x), if we suppose $x=0$, the area PON, or $\psi(x)$ will $=0$;

and, therefore, in this case (C) must $=0$; $\therefore$ C always $=0$; hence, in general,

$$\psi(x) = \frac{4a^{\frac{1}{2}}x^{\frac{3}{2}}}{3} = \frac{2x}{3}.2a^{\frac{1}{2}}x^{\frac{1}{2}} = \frac{2xy}{3},$$

or the area of the parabola $PON = \frac{2}{3}$. rectangle $PN.NO$.

From this example we may see, that the areas of all curves, in which it is possible to express (y) in terms of (x), might be found, if we could always find a function of (x), whose differential $= y\,dx$. The process, by which such functions are found, is called integration, and the investigation of it belongs to the integral, and not to the differential, calculus.



Cor. 2. If AOP be the sector of a circle, whose radius $OA = a$, versed sine $AN = x$,

then $d.(AOP) = \dfrac{a^2dx}{2\sqrt{2ax - x^2}}$.

Let $y = PN$,

then area APO = area APN + triangle PNO;

$$\therefore\ d.APO = y\,dx + d.\left\{\frac{y.(a-x)}{2}\right\}$$

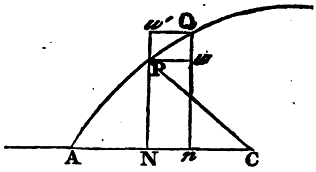
$$= y\,dx + \frac{dy.(a-x) + y\,dx}{2}$$

$$= \frac{y\,dx + a\,dy - x\,dy}{2},$$

but $y = \sqrt{2ax - x^2}$; $\therefore\ dy = \dfrac{a-x}{\sqrt{2ax-x^2}}$;

$$\therefore\ d.APO = \frac{1}{2}.\left\{\sqrt{2ax-x^2}.dx + \frac{(a-x)^2dx}{\sqrt{2ax-x^2}}\right\}$$

$$= \frac{dx}{2}.\frac{a^2}{\sqrt{2ax-x^2}} = \frac{a^2dx}{2\sqrt{2ax-x^2}}.$$



Cor. 3. If we suppose the curve APQ in the figure to this proposition to represent the circular end of a cylinder, whose length $= (c)$, and if we represent the content of that part of the cylinder, which stands upon the curved area APN, by $\psi'(x)$, we have that part of the cylinder, which stands upon the area $QPNn$

$$= \psi'.(x+h) - \psi'(x)$$

$$= f\psi'.(x)\,h + f^2.\psi'(x)\,\frac{h^2}{1\,.\,2} + \&c. \qquad (C).$$

But the content of a parallelopiped standing upon the rectangle Pn, and having its length = length of the cylinder $= c$, will equal $cyh = c\,.\,\phi(x)\,h$.

And the content of a parallopiped standing upon the rectangle QN, and having its length $= c$, will equal

$$c\,.\,\phi(x+h)\,.\,h$$

$$= c\,.\,\phi(x)\,.\,h + c\,.f.\,\phi(x)\,h^2 + \&c. \qquad (D).$$

Now the content of that part of the cylinder, which stands upon the area $QPNn$, lies between the contents of these two parallelopipeds; and, therefore, the series (C) will lie between the quantity $c\phi(x)\,h$, and the series (D), and, consequently,

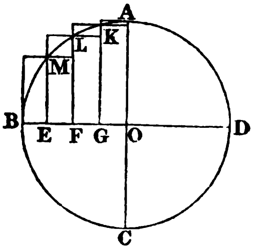
$$f.\psi'(x) = c\,.\,\phi(x) = c\,.f.\,\psi(x) = f\,.\,c\,\psi(x)\,;$$

that is, $\psi'(x)$ and $c \, . \, \psi(x)$ are two functions of (x), which have the same differential coefficient; and, therefore, the difference between them is constant; but, when $x = 0$, they both $= 0$; and, therefore, their difference then $= 0$; and, consequently, their difference always $= 0$; hence, $\psi'(x)$ always $= c \, . \, \psi(x)$. Now, supposing (x) to become $=$ radius of the circle, $\psi'(x)$ then $=$ one fourth of the content of the cylinder, and $\psi(x) =$ area of the quadrant; hence,

$$\frac{\text{content of cylinder}}{4} = c \times \text{area of the quadrant};$$

$\therefore$ content of the cylinder $=$ its length $\times$ area of its circular end; as has been proved, (Prop. 6.).

Cor. 4. An approximate value of the area of any curve may be found by the differential calculus only.



Let *BAO* be any curve, let $BE = x$, $EM = y = \phi(x)$, and area $BEM = \psi(x)$, and let $EF = h$.

then $\psi(x+h) - \psi(x)$ or area *EMLF*

$$= f \, . \, \psi(x) \, . \, h + f^2 . \, \psi(x) \frac{h^2}{1 \, . \, 2} + \&c.$$

$$= \phi(x) \, . \, h + f \, . \, \phi(x) \frac{h^2}{1 \, . \, 2} + \&c.;$$

hence, having found the differential coefficients of $\phi(x)$, we may, by taking (h) so small, that the series converges rapidly, calculate, to any required degree of accuracy, the area *EMLF*.

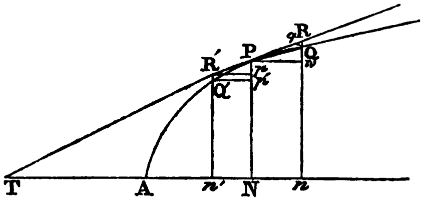
Again, making $x = BF$, and $h = FG$, we may, in the same manner, calculate the area *FLKG*; and thus, if the area *BAO* be divided into any number of parts by the parallel ordinates *ME*, *LF*, &c., the areas, included between these ordinates may be calculated; and, therefore, the whole area may be found.

There are many artifices, by which this calculation may be facilitated; a knowledge of these artifices is necessary to such as intend to make Mathematics the study of their life, but is useless to those, who only consider them as the means, by which their minds may be strengthened, and thus better qualified to enter upon the studies of after life.

Prop. XLV.

The length of a curve, corresponding to any abscissa, has its differential coefficient at any point equal to the secant of the angle, which the tangent to the curve at that point makes with the horizontal axis.

Let *APQ* be the curve, and (having made the same construction as before), join *PQ*, let the arc



$AP = s = \phi'(x)$.

Then the arc $PQ = \phi'(x+h) - \phi'(x)$

$$= f\phi'(x)h + f^2.\phi'(x)h^2 + \&c. \quad (A).$$

Now, (by Axioms, p. 113.), the arc *PQ* is less than

$(PR+RQ)$ and greater than the chord PQ. But (chord $PQ+RQ$) is greater than PR; therefore, the chord PQ is greater than $(PR-RQ)$; hence, *a fortiori*, the arc PQ is greater than $(PR-RQ)$; and, consequently, it lies between $(PR+RQ)$ and $(PR-RQ)$,

$$\text{but, } PR = P\pi . \sec RP\pi = h . \sec \theta$$

(θ being the angle PTN).

Also, $Rn = \phi(x) + \tan\theta . h = \phi(x) + f.\phi(x).h$,

and $Qn = \phi(x+h)$

$$= \phi(x) + f.\phi(x).h + f^2.\phi(x).\frac{h^2}{1.2} + \&c.;$$

also $QR = Rn - Qn$

$$= -f^2.\phi(x).\frac{h^2}{1.2} - f^3.\phi(x).\frac{h^3}{1.2.3} - \&c.;$$

$$\text{hence, } PR+RQ = \sec\theta.h - f^2.\phi(x)\frac{h^2}{1.2} - \&c. \quad (B),$$

$$\text{and } PR-RQ = \sec\theta.h + f^2.\phi(x)\frac{h^2}{1.2} + \&c. \quad (C).$$

But, since the arc PQ lies between $(PR+RQ)$ and $(PR-RQ)$, the series (A) lies between the series (B) and (C); and, therefore, (by Prop. 13. Cor. 1.), $f.\phi'(x) = \sec\theta$.

Cor. 1. Since $\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{dy^2}{dx^2}$, we have

$$\overline{f.\phi'(x)}|^2 \text{ or } \frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2};$$

and hence, $ds^2 = dx^2 + dy^2$;

$$\therefore ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \frac{dy^2}{dx^2}} = dy\sqrt{1 + \frac{dx^2}{dy^2}}.$$

COR. 2. Hence, if (s) be the arc of a circle whose radius is (a), and versed sine (x), we have

$$ds = \frac{adx}{\sqrt{2ax - x^2}},$$

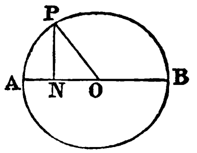
$$\text{for } y = \sqrt{2ax - x^2}; \therefore dy = \frac{(a-x).dx}{\sqrt{2ax - x^2}};$$

$$\therefore dy^2 + dx^2 \text{ or } ds^2 = dx^2\left\{\frac{(a-x)^2}{2ax - x^2} + 1\right\}$$

$$= \frac{a^2dx^2}{2ax - x^2};$$

$$\therefore ds = \frac{adx}{\sqrt{2ax - x^2}}.$$

COR. 3. Hence, the area of a circle $APB =$ the rectangle contained by a straight line equal to its semi-circumference APB, and the radius AO.



Let $AO = a$, $AN = x$, $AP = s$, and the area of the sector $PAO = S$, then, (by Cor. to former Prop.),

$$dS = \frac{a^2dx}{2\sqrt{2ax - x^2}} = \frac{a}{2}.ds = d.\frac{as}{2}.$$

Hence, S and $\frac{as}{2}$, are two functions of (x), which have the same differential coefficient; and, therefore, the difference between them is constant; but, when $x = 0$, S and (s) both $= 0$; and, therefore, the difference then $=0$; and, therefore, S and $\frac{as}{2}$ must always be equal. Now, suppose (s) to be the semi-circumference APB, then S equal half the area of the circle; hence,

$$\frac{\text{area of circle}}{2} = \frac{a \times \text{semi-circumference } APB}{2},$$

or area of circle = radius × semi-circumference.

Cor. 4. Since $\pi a =$ semi-circumference APB, the area of the circle $= \pi a^2$, (as was proved, Prop. 39.).

Cor. 5. If $a=1$, the area of the circle $=\pi$; that is, the same number, which represents the length of the semi-circumference of a circle, whose radius is unity, will also represent its area.

Cor. 6. Hence, the content of a cylinder, whose length is (c), and radius $(a) = \pi a^2 c$, for πa^2 represents the area of its semi-circular end.

Ex. 1. To find the length of a curve, whose equation is

$$y = \sqrt{2ax - x^2} + a\,.\,\text{ver. sin}^{-1}\frac{x}{a},$$

differentiating this equation, we have

$$dy = \frac{(a-x)\,dx}{\sqrt{2ax-x^2}} + \frac{a\,dx}{\sqrt{2ax-x^2}}$$

$$= \frac{(2a-x)\,dx}{\sqrt{2ax-x^2}} = \sqrt{\frac{2a-x}{x}}\,.\,dx;$$

$$\therefore\ dx^2 + dy^2 = dx^2\left\{1 + \frac{2a-x}{x}\right\};$$

$$\therefore\ ds^2 = dx^2 . \frac{2a}{x};\quad \therefore\ ds = \sqrt{2a}\,\frac{dx}{x^{\frac{1}{2}}},$$

$$\text{but } d.2x^{\frac{1}{2}} = \frac{dx}{x^{\frac{1}{2}}};\quad \therefore\ ds = \sqrt{2a}.d.2x^{\frac{1}{2}}$$

$$= d.2\sqrt{2ax}.$$

Hence, (s) and $2\sqrt{2ax}$ are two functions of (x), which have the same differential; and, consequently, their difference is constant; but when $x=0$, if we suppose $s=0$, their difference will then $=0$; and, therefore, it will always $=0$, or $s=2\sqrt{2ax}=$ twice the chord of an arc, whose versed sine is (x), and radius $=a$. When $x=2a$, (s) becomes the length of the curve, and $=4a$. The length of that part of the curve, which lies on the other side of the axis, along which (x) is measured, also $=4a$; and, therefore, the whole length $=8a$. This curve is the cycloid.

Ex. 2. To find the length of an arc of an ellipse, whose axis-major $=2a$, and whose eccentricity $=e$.

$$\text{Here } y = \frac{b}{a}\sqrt{a^2-x^2};\quad \therefore\ dy = \frac{-bxdx}{a\sqrt{a^2-x^2}};$$

$$\therefore\ dx^2+dy^2 = dx^2\left\{1+\frac{b^2x^2}{a^2(a^2-x^2)}\right\}$$

$$= dx^2.\left\{\frac{a^4-(a^2-b^2)\,x^2}{a^2.(a^2-x^2)}\right\};$$

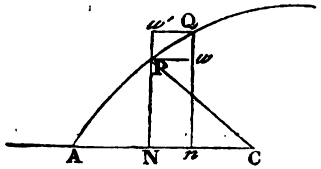
$$\therefore\ ds^2 = \left\{\frac{a^2-\dfrac{a^2-b^2}{a^2}.x^2}{a^2-x^2}\right\}.dx^2 = \frac{(a^2-e^2x^2)\,dx^2}{a^2-x^2};$$

$$\therefore\ ds = dx\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}};$$

therefore $s =$ a function of (x), whose differential is $dx\sqrt{\dfrac{a^2-e^2x^2}{a^2-x^2}}$. This function cannot be expressed in finite algebraical terms. If $e=0$, the ellipse becomes a circle, and $ds = \dfrac{a\,dx}{\sqrt{a^2-x^2}}$, as before proved.

Prop. XLVI.

If a curve APQ *revolves about its abscissa* ANn *as an axis, the differential coefficient of the solid content, traced out by the revolution of the area* $APN = \pi . PN^2$.



Let $\psi'(x)$ represent this solid content, and, using the same figure as before, complete the rectangles *Pn*, *QN*; then, supposing the curve to revolve, the content, generated by the revolution of the curved area *QPNn*

$$= \psi'(x+h) - \psi'(x)$$

$$= f.\psi'(x)h + f^2.\psi'(x).\frac{h^2}{1.2} + \&c. \qquad (A).$$

The cylinder generated by the revolution of the rectangle *Pn* equals $\pi y^2 h$ or $\pi.\{\phi(x)\}^2.h$.

And the cylinder generated by the rectangle $QN = \pi.Qn^2.h$.

$$= \pi.\{\phi(x+h)\}^2.h$$

$$= \pi.\{\phi(x) + f.\phi(x)\,h + f^2.\phi(x)\frac{h^2}{1.2} + \&c.\}^2 h$$

$$= \pi.\{[\phi(x)]^2.h + 2.\phi(x).f.\phi(x).h^2 + \&c.\} \quad (B).$$

Now the content, generated by the curve, lies between these two cylinders; $\therefore$ the series (A) lies between the quantity $\pi\{\phi(x)\}^2 h$, and the series (B); hence, (by Prop. 13. Cor. 2.)

$$f.\psi'(x) = \pi.\{\phi(x)\}^2. = \pi y^2 = \pi.PN^2.$$

COR. 1. Hence, $\dfrac{d.\psi'(x)}{dx} = \pi y^2$;

$$\therefore\ d.\psi'(x) = \pi y^2 dx.$$

Ex. 1. If the curve is a parabola, we have $y^2 = 4ax$; hence,

$$\pi y^2 dx = 4\pi a x dx = 2\pi a d.x^2 = d.2\pi a x^2,$$

that is, the content of the solid, generated by the revolution of the parabola, and the quantity $2\pi a x^2$, are two functions of (x), which have the same differential, and, therefore, the difference between them is constant; but this difference $=0$, when $x=0$; and, therefore, the content of the solid $= 2\pi a x^2 = \dfrac{\pi y^2 x}{2} = \dfrac{1}{2}$ content of a cylinder, whose length = length of the parabola, and whose radius = radius of its base.

This content might have been found merely by differentiation. For the content generated by the revolution of the curved area $QPNn$

$$= f\psi'(x).h + f^2.\psi'(x)\frac{h^2}{1.2} + \&c.,$$

but, in this case, $f\psi'(x) = \pi y^2 = 4\pi a x$;

hence, $f^2\psi'(x)$ or $\dfrac{d.f.\psi'(x)}{dx} = 4\pi a$, a constant quantity; and, therefore, all the other differential coefficients vanish, (by Prop. 19.) Hence,

$$\text{the solid due to } QPNn = 4\pi a x h + 4\pi a . \frac{h^2}{1 . 2},$$

which being true, whatever be the value of (x), suppose $x = 0$, then $h = An$, and the solid generated by the area $APQn = 2\pi a h^2$, and writing (x) for (h), the solid generated by the area, which corresponds to the abscissa $(x) = 2\pi a x^2 = \frac{\pi y^2 x}{2}$, as before.

Ex. 2. To find the content of a sphere, by differentiation.

Here, $y^2 = 2ax - x^2$; hence, $f\psi'(x) = \pi . (2ax - x^2)$;

$$\therefore f^2 . \psi'(x) = \pi (2a - 2x), \text{ and } f^3 . \psi'(x) = -2\pi.$$

Hence, solid due to $QPNn$

$$= \pi \left\{ (2ax - x^2) h + (2a - 2x) . \frac{h^2}{1 . 2} - 2 . \frac{h^3}{1 . 2 . 3} \right\}.$$

Now, let $x = 0$, then (h) becomes $= An$, and the area $QPNn$ becomes the area $APQn$.

Hence, the content due to the area $APQn$

$$= \pi \left(a h^2 - \frac{h^3}{3} \right),$$

and making $h = 2a$, we have the whole content of the sphere

$$= \pi \left(4a^3 - \frac{8a^3}{3} \right) = \frac{4\pi a^3}{3}.$$

Now, if a cylinder be described about this sphere, the radius of this cylinder will $= a$, and its length $= 2a$; and, therefore, its content $= 2\pi a^3$.

Hence, content of sphere : content of circumscribing cylinder :: 2 : 3.

Ex. 3. To find the content of a cone by differentiation.

Let (c) be the altitude of the cone, and (a) the radius of its base, then the equation to it is

$$y = \frac{a}{c}x; \quad \therefore y^2 = \frac{a^2x^2}{c^2};$$

$$\text{hence, } f.\psi'(x) = \frac{\pi a^2 x^2}{c^2}; \quad \therefore f^2\psi'(x) = \frac{2\pi a^2 x}{c^2};$$

$$\text{and } f^3\psi(x) = \frac{2\pi a^2}{c^2};$$

hence, while (x) increases by (h), the quantity, by which $\psi'(x)$, or the content of the cone increases, is

$$\frac{\pi a^2 x^2}{c^2}.h + \frac{2\pi a^2 x}{c^2}.\frac{h^2}{1.2} + \frac{2\pi a^2}{c^2}.\frac{h^3}{1.2.3}.$$

Now, to find the whole content, suppose (x) to increase from 0, till it$=c$, then making $x=0$, and $h=c$, we have

$$\text{the whole content} = \frac{2\pi^2 a^2}{c^2}.\frac{c^3}{1.2.3} = \frac{\pi a^2 c}{3}.$$

Hence, the content of a cone is equal to one-third of the content of its circumscribing cylinder.

Prop. XLVII.

If a curve revolves about its abscissa ANn, *as an axis, and if the arc* AP$=$s, *and the surface generated by the revolution of it$=$*S, *there results the equation*

$$d.S = 2\pi.PN.ds.$$

Let Qn (See Fig. Prop. 46.) be an ordinate parallel to PN, and let $Nn = h$. Then the surface generated by the revolution of the arc PQ, will equal the series

$$\frac{dS}{dx}.h + \frac{d^2S}{dx^2}.\frac{h^2}{1.2} + \frac{d^3S}{dx^3}.\frac{h^3}{1.2.3} + \&c. \qquad (A).$$

The curved surface of a cylinder, whose length equals the arc PQ, and whose radius $= PN = y$, is, (by Prop. 37. Cor. 4.) equal to $2\pi y \times$ arc PQ.

$$\text{But the arc } PQ = \frac{ds}{dx}.h + \frac{d^2s}{dx^2}.\frac{h^2}{1.2} + \&c. \qquad (M).$$

Hence, the surface of this cylinder is equal to the series

$$2\pi y\left(\frac{ds}{dx}.h + \frac{d^2s}{dx^2}.\frac{h^2}{1.2} + \&c.\right) \qquad (B).$$

Also the curved surface of a cylinder, whose length equals the arc PQ, and whose radius equals Qn, is equal to $2\pi.Qn.$arc PQ.

$$\text{But } Qn = y + \frac{dy}{dx}.h + \frac{d^2y}{dx^2}.\frac{h^2}{1.2} + \&c. \qquad (N).$$

Hence, the curved surface of this latter cylinder is equal to $2\pi \times$ the product of the two series (M) and (N), or

$$2\pi\left\{\frac{yds}{dx}.h + \left(\frac{2ds}{dx}.\frac{dy}{dx} + \frac{yd^2s}{dx^2}\right).\frac{h^2}{1.2} + \&c.\right\}. \qquad (C).$$

Now, since (by the third Axiom), the surface of the solid lies between these two cylinders, the series (A) must lie between the series (B), and the series (C); hence,

$$\frac{dS}{dx} = \frac{2\pi y ds}{dx}; \text{ and } \therefore d.S = 2\pi y ds.$$

Cor. 1. Since $ds = \sqrt{dy^2 + dx^2}$, we have

$$dS = 2\pi y . \sqrt{dy^2 + dx^2}.$$

Ex. 1. To find the surface of a sphere.

Here $ds = \dfrac{adx}{\sqrt{2ax - x^2}} = \dfrac{adx}{y}$; $\therefore yds = adx$;

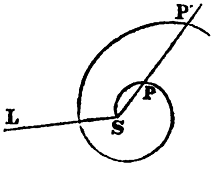
$$\therefore d.S \text{ (which } = 2\pi y ds) = 2\pi a dx = d.(2\pi ax);$$

hence, S and $2\pi ax$ are two functions of (x) which have the same differential coefficient; and, therefore, their difference is constant; but when $x = 0$, $S = 0$; and, therefore, their difference then $= 0$; hence, it always $= 0$; and, consequently, $S = 2\pi ax$, and when $x = 2a$, the whole surface becomes $= 4\pi a^2$; hence, the whole surface of a sphere equal four times the area of one of its great circles.

SECT. VI.

ON SPIRALS.

THERE are many curves, the form and properties of which are not determined by an equation between two co-ordinates, referred to rectangular axes, but by an equation expressing the relation which subsists between the length of a straight line, drawn from a given point, and the angle, which this straight line makes with another given straight line, drawn from the same point. Such curves are called Spirals.



Let *S* be a given point, *SL* a given straight line, then, if other straight lines, as *SP*, be drawn from *S*, whose lengths are functions of the angles *LSP*, which they make with *SL*, the locus of the points *P* will be a curve, and this curve is, by the definition, a spiral.

Since *SP* may be supposed to revolve round *S*, we may imagine it to complete a revolution, and to be again brought into the position *SP*; then producing *SP*, if necessary, and taking *SP′*, the same function of the angle $(LSP + 2\pi)$, that *SP* is of the angle *LSP*, *P′* will be another point in the curve. Again, supposing *SP* to complete another revolution, we may find other points in *SP* produced, which belong to the spiral.

In some spirals, there is no limit to the number of revolutions, which *SP* must make, in order to complete the curve, in others, the angle *LSP* never exceeds a certain limit.

Def. The point *S* is generally termed the pole.

Def. The line *SL* is generally termed the initial line.

Def. The line *SP* is termed the radius vector of the spiral, and is generally represented by the Greek letter (ρ).

Def. The angle *LSP* is termed the spiral angle, and is generally represented by the Greek letter (θ).

Def. The equation, which expresses the relation between (ρ) and (θ), is termed the equation to the spiral, and may be represented under the general form

$$\rho = \phi(\theta),$$

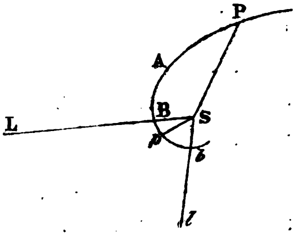
$\phi(\theta)$ being the symbol, which expresses the function, that (ρ) is of (θ).

It is manifest, that since *SP* is a function of the angle *LSP*, the area of the spiral, its length, its perpendicular upon the tangent, &c. are all functions of the angle *LSP*; and as such they will always be regarded in the following propositions.

Whenever in this, and in the eighth Section, we speak of the differential coefficients of any of these quantities, we mean their differential coefficients, when considered as functions of the angle *LSP* or (θ).

PROP. XLVIII.

If $\rho=\phi(\theta)$ *be the equation to any spiral, and if, when* R *and* $(-\Theta)$, *(the latter being negative,) are substituted in this equation for* (ρ) *and* (θ) *respectively, the terms of the equation are fulfilled; the angle, which the radius vector* (R) *makes with the initial line, is equal to* (Θ), *but lies on the contrary side of the initial line to that, on which the positive angles are measured.*



Let pAP be a spiral, S its pole, SL its initial line, SP any radius vector $=\rho$, the angle $LSP=\theta$, then by the supposition, $\rho=\phi(\theta)$. At the point S, make the angle $LSp=\Theta$, and $Sp=R$, then, what we have to prove is, that (p) is a point in the spiral.

At the point S make any angle LSl greater than Θ, and suppose it $=a$, and let the angle $lSP=\theta'$,

$$\text{then } \theta'=\theta+a, \quad \text{and } \therefore\ \theta=\theta'-a,$$

$$\text{but } \rho=\phi(\theta); \text{ hence, } \rho=\phi(\theta'-a).$$

This equation will represent the relation between the radius vector SP, and the angle lSP, when this angle is greater than lSL, or, in other words, it will be the equation to that part of the spiral, which lies on the positive side of SL, when Sl is taken for the initial line. But, since the spiral is a regular curve, it is determined by the same equation throughout; hence, the equation

$$\rho=\phi(\theta'-a).$$

will represent the relation between the radius vector, and the angle it makes with Sl, in that part of the spiral, which lies between SL and Sl, where the angle (θ') is positive, but less than (a). Now in the equation

$$\rho = \phi(\theta' - a)$$

substitute the angle lSp, which $= a - \Theta$ for (θ'), then we have

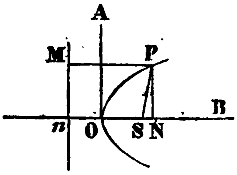
$$\rho = \phi(a - \Theta - a) = \phi(-\Theta) = R.$$

Hence, when the spiral is referred to Sl as its initial line, the radius vector, which corresponds to the angle $lSp = R = Sp$; and, therefore, (p) is a point in the spiral.

Cor. 1. Hence, if a point (p') be taken in the line SP, such that Sp' is the same function of $-(2\pi - \theta)$, that SP is of (θ), p' is a point in the spiral.

It is manifest, that all regular curves may be considered either as spirals, or as curves referred to rectangular axes, since the difference between spirals and other curves does not arise from the nature of the curves themselves, but from the methods we make use of, to ascertain their properties.

Thus, if a parabola be considered as a common curve, its equation is $y^2 = 4ax$, where (x) is the abscissa, measured along the principal diameter as an horizontal axis, from the vertex as first point, and (y) the ordinate to this abscissa; but if the parabola be considered as a spiral, $\rho = a \sec^2 \frac{\theta}{2}$, (ρ) being the radius vector measured from the focus, and (θ) the angle which is contained between SP, and the line joining the focus and the vertex.



Let OP be a parabola, S its focus, and also its pole, SO the initial line, where O is the vertex; then, if $SO=a$, $PN=y$, $ON=x$ $SP=\rho$, and the angle $OSP=\theta$, there will result these equations,

$$y^2=4ax,$$

$$y=\rho\,.\sin\theta,$$

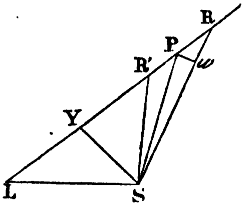
$$x=a-\rho\,.\cos\theta;$$

whence, eliminating (x) and (y), we have

$$\rho=a\,.\sec^2\frac{\theta}{2}\,.$$

Prop. XLIX.

Having given the position of the pole, to find the differential coefficients of the radius vector of a straight line, when considered as a spiral.



Let LP be the given straight line, considered as a spiral, S the pole, SY a perpendicular upon LP, SR', SP, SR three radii vectores, having the angle $R'SP=$ the angle RSP.

Suppose SY the initial line, and let the spiral angle $YSP=\theta$, and each of the angles RSP, $R'SP$ $=\iota$; then SP or (ρ) manifestly

$$=SY\times\sec\theta=p\,.\sec\theta,\ (\text{if } SV=p);$$

$$\text{hence, } SR=p\,.\sec(\theta+\iota)$$

$$=p\left\{\sec\theta+\frac{d\,.\sec\theta}{d\theta}\,.\,\iota\right.$$

$$\left.+\frac{d^2(\sec\theta)}{d\theta^2}\cdot\frac{\iota^2}{1\,.\,2}+\&c.\right\}$$

$$=p\,.\,\{\sec\theta+\sec\theta\,.\,\tan\theta\,.\,(\iota)$$

$$+\sec\theta\,.\,(2\tan^2\theta+1)\,\frac{\iota^2}{1\,.\,2}+\&\text{c.}\}$$

as appears by actual differentiation; now, if the coefficients of the different powers of (ι), thus found, be represented by A_1, A_2, A_3, &c. respectively, we have

$$SR=\rho+A_1p\,(\iota)+A_2p\,.\,\frac{\iota^2}{1\,.\,2}+A_3\,.\,p\,.\,\frac{\iota^3}{1\,.\,2\,.\,3}+\&\text{c.}$$

And similarly,

$$SR'=\rho-A_1p\,(\iota)+A_2p\,.\,\frac{\iota^2}{1\,.\,2}-A_3\,.\,p\,.\,\frac{\iota^3}{1\,.\,2\,.\,3}+\&\text{c.}$$

COR. 1. Hence also the differential coefficients of the length of the straight line YP may be found; for $PY=p\,.\,\tan\theta$; hence,

$$RY=p\,.\,\tan\,(\theta+\iota)$$

$$=p\,.\left\{\tan\theta+\frac{d\,.\tan\theta}{d\theta}\,(\iota)+\frac{d^2\tan\theta}{d\theta^2}\,.\,\frac{\iota^2}{1.2}+\&\text{c.}\right\}.$$

$$=p\,.\left\{\tan\theta+\sec^2\theta\,.\,(\iota)+2.\tan\theta.\sec^2\theta\,.\,\frac{\iota^2}{1.2}+\&\text{c.}\right\},$$

as appears by actual differentiation, and representing the coefficients of the different powers of (ι) by B_1, B_2, B_3, &c., we have

$$YR=p\left\{\tan\theta+B_1\,(\iota)+B_2\,.\,\frac{\iota^2}{1.2}+B_3\,.\,\frac{\iota^3}{1\,.\,2\,.\,3}+\&\text{c.}\right\}.$$

And, similarly,

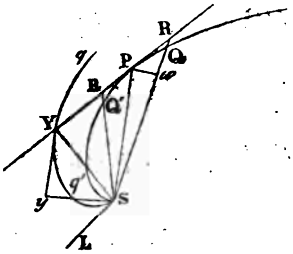
$$YR'=p\,.\left\{\tan\theta-B_1\,(\iota)+B_2\,.\,\frac{\iota^2}{1.2}-B_3\,.\,\frac{\iota^3}{1.2.3}+\&\text{c.}\right\},$$

Cor. 2. Hence, $PR = pB_1\,(\iota) + pB_2 \cdot \frac{\iota^2}{1\,.\,2}$

$$+\,p\,.\,B_3\,.\,\frac{\iota^3}{1\,.\,2\,.\,3} + \&c.$$

Prop. L.

To draw a tangent to a spiral.



Let SPQ be a spiral, YPR a tangent at any point P, SQR, $SQ'R'$, two radii vectores produced to meet the tangent in R and R', SY a perpendicular upon the tangent, SL the initial line. Let the angle $LSP = \theta$, the angle $RSP =$ the angle $R'SP = \iota$, the angle $YSP = \theta'$, $SP = \rho$, and let the equation to the spiral be $\rho = \phi\,(\theta)$;

then $SQ = \phi\,.\,(\theta + \iota)$

$$= \phi\,(\theta) + f.\,\phi\,(\theta)\,.\,(\iota) + f^2.\,\phi\,(\theta)\,.\,\frac{\iota^2}{1\,.\,2} + \&c.$$

Similarly, $SQ' = \phi\,(\theta) - f.\phi\,(\theta)\,.\,\iota + f^2.\phi\,(\theta)\,.\,\frac{\iota^2}{1.2} - \&c.$

$$\text{But } SR = \rho + A_1 p\,.\,\iota + A_2 p\,.\,\frac{\iota^2}{1\,.\,2} + \&c.$$

$$\text{and } SR' = \rho - A_1 p\,.\,\iota + A_2 p\,.\,\frac{\iota^2}{1\,.\,2} - \&c.$$

Now, since YP is a tangent to the spiral, if SR be greater than SQ, SR' is greater than SQ', and if less, less.

First, let SR be greater than SQ, then $SR - SQ$ is positive, but $SR - SQ$ is represented by the series

$$\{A_1p-f.\phi(\theta)\}.\iota+\{A_2p-f^2.\phi(\theta)\}.\frac{\iota^2}{1.2}+\&c.$$

Hence, this series is positive; and, consequently, A_1p cannot be less than $f.\phi(\theta)$.

Again, $SR'-SQ'$ is positive, but it is represented by the series

$$\{f.\phi(\theta)-A_1p\}.\iota+\{A_2p-f^2.\phi(\theta)\}.\frac{\iota^2}{1.2}-\&c.$$

Hence, this series is positive; and, therefore, A_1p cannot be greater than $f.\phi(\theta)$; and, consequently,

$$f.\phi(\theta) \text{ or } \frac{d\rho}{d\theta}=A_1p=\tan\theta'.\sec\theta'p, \text{ (Prop. 49.)}$$

$$=\rho.\tan\theta';$$

$$\therefore\ \frac{d\rho}{\rho d\theta}=\tan\theta'=\text{co-tan } \angle SPY.$$

Hence, if at the point P, we draw a straight line PY, making an angle with the radius vector, whose co-tangent $=\frac{d\rho}{\rho\, d\theta}$, this line will be a tangent to the spiral, (provided the spiral admits a tangent at that point).

Cor. 1. If SY the perpendicular upon the tangent $=p$, we have

$$p=\rho.\cos\theta'=\frac{\rho}{\sqrt{1+\tan^2\theta'}}=\frac{\rho^2 d\theta}{\sqrt{\rho^2 d\theta^2+d\rho^2}}.$$

Ex. 1. To draw a tangent to the spiral of Archimedes.

Here $\rho=a\theta$, (a being constant); hence,

$$\therefore \frac{d\rho}{\rho . d\theta} = \frac{a}{\rho};$$

hence, the tangent of the angle *SPY* is equal $\frac{\rho}{a}$.

Also $p = \frac{\rho^2 d\theta}{\sqrt{\rho^2 d\theta^2 + d\rho^2}} = \frac{\rho^2}{\sqrt{\rho^2 + a^2}}$.

Ex. 2. To draw a tangent to the hyperbolic spiral.

Here $\theta = \frac{a}{\rho}$; $d\theta = \frac{-a d\rho}{\rho^2}$; $\therefore \frac{d\rho}{\rho . d\theta} = -\frac{\rho}{a}$;

hence, the tangent of the angle $SPY = -\frac{a}{\rho}$.

Ex. 3. To draw a tangent to the parabola.

Here $\rho = a . \sec^2 \frac{\theta}{2}$; $\therefore \frac{d\rho}{\rho . d\theta} = \tan \frac{\theta}{2}$.

Hence it appears, that the perpendicular upon the tangent to any point bisects the spiral angle to that point; also,

$$p = \frac{\rho^2 d\theta}{\sqrt{\rho^2 d\theta^2 + d\rho^2}} = \sqrt{\rho a}.$$

Ex. 4. To draw a tangent to the logarithmic spiral.

Here $\rho = a^{\theta}$; $\therefore$ hyp. log. $\rho = \theta$. hyp. log. a;

$$\therefore \frac{d\rho}{\rho d\theta} = \text{hyp. log. } (a).$$

Hence, the tangent of the angle $SPY = \frac{1}{\text{hyp. log. } (a)}$, and is constant; and, therefore, all the radii vectores are equally inclined to their corresponding tangents.

Hence, this curve is called the equiangular spiral.

Also, $p = \frac{\rho^2 d\theta}{\sqrt{\rho^2 d\theta^2 + d\rho^2}} = \frac{\rho^2}{\sqrt{\rho^2 + \rho^2 . \{\text{hyp. log. } (a)\}^2}}$

$$= \frac{\rho}{\sqrt{1 + \{\text{hyp. log. } (a)\}^2}};$$

hence, the ratio of SY to SP in this curve is a constant ratio.

Cor. 2. Hence, $\frac{1}{p} = \frac{\sqrt{\rho^2 d\theta^2 + d\rho^2}}{\rho^2 d\theta}$;

$$\therefore \frac{1}{p^2} = \frac{1}{\rho^2} + \frac{d\rho^2}{\rho^4 d\theta^2}.$$

Cor. 3. Hence, $d\theta = \frac{p\,d\rho}{\rho\sqrt{\rho^2 - p^2}}$; and, consequently, if we know (p) in terms of (ρ), we may, by integration, find the value of θ.

Cor. 4. Since $f.\phi(\theta) - A_1 p = 0$, $SR - SQ$ or QR is equal to series

$$\{A_2 p - f^2.\phi(\theta)\}.\frac{\iota^2}{1.2} + \{A_3 p - f^3\phi(\theta)\}.\frac{\iota^3}{1.2.3} + \&c.$$

$$\text{or, } R_2.\frac{\iota^2}{1.2} + R_3.\frac{\iota^3}{1.2.3} + \&c. \qquad (D),$$

if we assume R_2, R_3, &c. equal to the coefficients of $\frac{\iota^2}{1.2}$, $\frac{\iota^3}{1.2.3}$, &c. respectively; and, similarly,

$$SR' - SQ' \text{ or } Q'R' = R_2.\frac{\iota^2}{1.2} - R_3\frac{\iota^3}{1.2.3} + \&c. \qquad (E).$$

Cor. 5. Hence, a spiral admits only one tangent at one point. For it has been shown, that if PY be a tangent, the angle SPY must be one, whose co-tangent $= \frac{d\rho}{\rho \, . \, d\theta}$; and, therefore, no other straight line, which makes with SP a greater or a less angle, can be a tangent.

Cor. 6. Hence, a spiral does admit a tangent, when R_2 is a finite positive or negative quantity. For if R_2 be finite and positive, the value of (ι) may be so diminished, that the first term in each of the series (D) and (E) shall be greater than the sum of all the rest. Hence, both $SR - SQ$, and $SR' - SQ'$ will then be positive; consequently, SR and SR' will be greater than SQ and SQ' respectively; and, therefore, both Q and Q' will lie between the straight line YPR and the pole; and, therefore, YPR cannot cut the spiral at P; and, therefore, it is a tangent.

Cor. 7. A spiral will not admit a tangent at those points, where $A_2 p = f^2 . \phi(\theta)$ or $R_2 = 0$.

For, in that case,

$$SR - SQ = R_3 . \frac{\iota^3}{1.2.3} + R_4 . \frac{\iota^4}{1.2.3.4} + \&c. \quad (G),$$

and

$$SR' - SQ' = - R_3 . \frac{\iota^3}{1.2.3} + R_4 . \frac{\iota^4}{1.2.3.4} - \&c. \quad (K).$$

Now when R_3 is positive, if (ι) be sufficiently diminished, the series (G) will be positive; and, therefore, SR will be greater than SQ; and, therefore, the point R will lie without the spiral. Again, the series (K) will be negative; hence, SR' will be less than SQ'; and, therefore, R' will lie within the spiral. Hence, the line RPR' must cut the spiral. The same proof would have applied if R_3 had been negative.

Cor. 8. If, however, we have the two equations

$$A_2 p = f^2 . \phi(\theta) \text{ or } R_2 = 0,$$

$$\text{and } A_3 p = f^3 . \phi(\theta) \text{ or } R_3 = 0,$$

then the spiral will admit a tangent.

For, in that case, we shall have

$$SR - SQ = R_4 . \frac{\iota^4}{1.2.3.4} + R_5 . \frac{\iota^5}{1.2.3.4.5} + \&c. \quad (L),$$

and

$$SR' - SQ' = R_4 . \frac{\iota^4}{1.2.3.4} - R_5 . \frac{\iota^5}{1.2.3.4.5} + \&c. \quad (M).$$

Now, if R_4 be positive, and (ι) sufficiently diminished, both these series will be positive; and, therefore, SR and SR' will be greater than SQ and SQ' respectively; and, therefore, RPR' will be a tangent.

Cor. 9. In a similar way it may be shown, that, if the number of the coefficients

$$R_2, \ R_3, \ R_4, \ \&c.,$$

which vanish at any point, be odd, the curve does not admit a tangent, and that, if the number be even, it does.

Cor. 10. Hence, if two spirals, referred to the same pole, have at any point a common tangent, the first differential coefficients of their radii vectores must, at that point, be equal. For at that point, since their radii vectores, and their tangents coincide, (ρ) and $\frac{d\rho}{\rho . d\theta}$, and, consequently, $\frac{d\rho}{d\theta}$ must be the same in both.

Cor. 11. Hence, if two spirals, referred to the same pole, touch each other internally, $\frac{d\rho}{d\theta}$ is the same in both. For in this case the spirals have a common tangent.

Def. A spiral is said to be concave at any point when the parts of it, immediately adjacent to that point, lie between the pole, and the tangent to that point.

Def. A spiral is said to be convex at any point, when the tangent to that point lies between the pole and the arcs, immediately adjacent to that point.

Def. If, while the spiral angle increases, the spiral from being convex becomes concave, or from being concave becomes convex, the point, where the change takes place, is called a point of contrary flexure.

If, at any point, the radius vector of a spiral becomes its tangent, it is manifest, that, at this point, a change from convexity to concavity takes place; but, since the spiral angle does not increase during the change, this point is not a point of contrary flexure.

Prop. LI.

To find when a spiral is convex, and when concave.

Using the same figure, construction, and notation as before, it appears, that

$$SR - SQ = R_2 \cdot \frac{\iota^2}{1 \cdot 2} + R_3 \cdot \frac{\iota^3}{1 \cdot 2 \cdot 3} + \&c. \quad (D),$$

$$SR' - SQ' = R_2 \cdot \frac{\iota^2}{1 \cdot 2} - R_3 \cdot \frac{\iota^3}{1 \cdot 2 \cdot 3} + \&c. \quad (E).$$

Now if the curve be concave, Q lies between R and S, and SR is greater than SQ; and, therefore, the series (D) is positive; hence, R_2 the coefficient of the first term, must be positive; (unless it $=0$).

$$\text{But } R_2 = A_2 p - f^2 . \phi(\theta);$$

and, therefore, $f^2 \phi(\theta)$ or $\frac{d^2\rho}{d\theta^2}$ is less than $A_2 p$,

$$\text{but } A_2 p = p . \sec\theta' . (2\tan^2\theta' + 1)$$

$$= \rho . \left(\frac{2\,d\rho^2}{\rho^2 d\theta^2} + 1\right) = \frac{2\,d\rho^2}{\rho\, d\theta^2} + \rho;$$

hence, when the spiral is concave,

$$\frac{d^2\rho}{d\theta^2} \text{ is less than } \frac{2\,d\rho^2}{\rho\, d\theta^2} + \rho.$$

In the same manner it may be shown, that, when the spiral is convex,

$$\frac{d^2\rho}{d\theta^2} \text{ is greater than } \frac{2\,d\rho^2}{\rho\, d\theta^2} + \rho.$$

Cor. 1. Conversely. The spiral will be concave, or convex according as $\frac{d^2\rho}{d\theta^2}$ is less or greater than $\frac{2\,d\rho^2}{\rho\, d\theta^2} + \rho$.

Ex. 1. To find when the logarithmic spiral is concave, and when convex.

Here the equation is $\rho = a^\theta$;

$$\therefore \frac{d\rho}{d\theta} = A a^\theta, \text{ when } A = \text{hyp. log. } (a).$$

$$\text{Hence, } \frac{2\,d\rho^2}{\rho\, d\theta^2} + \rho = \frac{2A^2 a^{2\theta}}{a^\theta} + a^\theta = (2A^2 + 1)\, a^\theta,$$

$$\text{and } \frac{d^2\rho}{d\theta^2} = A^2 a^\theta.$$

Now, $A^2 a^\theta$ must always be less than $(2A^2+1)\,a^\theta$; consequently, $\frac{d^2\rho}{d\theta^2}$ must always be less than $\frac{2\,d\rho^2}{\rho d\theta^2}+\rho$; and, therefore, the curve will always be concave, and admit no point of contrary flexure.

Ex. 2. To find when the parabola is concave or convex.

Here the equation is

$$\rho = a.\sec^2\frac{\theta}{2} = a.(1+t^2), \ \left(\text{if } t = \tan\frac{\theta}{2}\right).$$

$$\text{Hence, } \frac{d\rho}{d\theta} = 2at.\frac{dt}{d\theta} = at.(1+t^2)$$

$$\left(\text{for } \frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}\right) = \rho.t;$$

$$\therefore\ \frac{d\rho^2}{\rho d\theta^2} = \rho t^2 = at^2.(1+t^2);$$

$$\therefore\ \frac{2\,d\rho^2}{\rho d\theta^2} + \rho = 2a.t^2(1+t^2) + a.(1+t^2)$$

$$= a\,(1+3t^2+2t^4).$$

$$\text{Also, } \frac{d^2\rho}{d\theta^2} = \frac{a}{2}.(1+t^2) + \frac{3at^2.(1+t^2)}{2}$$

$$= \frac{a}{2}.(1+4t^2+3t^4),$$

$$\text{Now } \frac{a}{2}.(1+4t^2+3t^4),$$

is always less than

$$a.(1+3t^2+2t^4);$$

consequently, $\frac{d^2\rho}{d\theta^2}$ is always less than $\frac{2\,d\rho^2}{\rho\,d\theta^2}+\rho$; and, therefore, the curve is always concave.

Ex. 3. To find when the Lituus, $\left(\text{a spiral whose equation is } \theta = \frac{a^2}{\rho^2}\right)$ is concave, and when convex.

$$\text{Here } \rho = \frac{a}{\theta^{\frac{1}{2}}}; \quad \therefore \frac{d\rho}{d\theta} = -\frac{a}{2\theta^{\frac{3}{2}}};$$

$$\therefore \frac{2\,d\rho^2}{\rho.d\theta^2} = \frac{a}{2\theta^{\frac{5}{2}}};$$

$$\text{and } \therefore \frac{2\,d\rho^2}{\rho\,d\theta^2} + \rho = a\left(\frac{1+2\theta^2}{2\theta^{\frac{5}{2}}}\right).$$

$$\text{Also, } \frac{d^2\rho}{d\theta^2} = \frac{3a}{4\theta^{\frac{5}{2}}}.$$

Now when the spiral is concave, $\frac{d^2\rho}{d\theta^2}$ is less than $\frac{2\,d\rho^2}{\rho\,d\theta^2}+\rho$; hence, $\frac{3a}{4\theta^{\frac{5}{2}}}$ is less than $a\left(\frac{1+2\theta^2}{2\theta^{\frac{5}{2}}}\right)$; and $\therefore$ $\frac{3}{4}$ is less than $\frac{1+2\theta^2}{2}$; hence, θ^2 is greater than $\frac{1}{4}$; and, therefore, (θ) is greater than $\frac{1}{2}$, and, conversely, if (θ) be greater than $\frac{1}{2}$, the spiral is concave; in the same manner it may be shown, that if (θ) be less than $\frac{1}{2}$, the spiral is convex. Hence it appears, that there is a point of contrary flexure, when $\theta = \frac{1}{2}$, or $\rho = a\sqrt{2}$.

Cor. 2. When the spiral is concave, p and ρ increase or decrease together; and, when the spiral is convex, as one increases, the other decreases. For we have seen, that

$$\frac{1}{p^2} = \frac{1}{\rho^2} + \frac{d\rho^2}{\rho^4 d\theta^2};$$

$$\therefore\ -\frac{dp}{p^3} = -\frac{d\rho}{\rho^3} + \frac{d\rho \, . \, d^2\rho}{\rho^4 d\theta^2} - \frac{2d\rho^3}{\rho^5 d\theta^2};$$

$$\therefore\ \frac{dp}{d\rho} = \frac{p^3}{\rho^4} \cdot \left(\rho + \frac{2d\rho^2}{\rho d\theta^2} - \frac{d^2\rho}{d\theta^2}\right) \qquad (A).$$

Now, by the former Corollary it appears, that when the spiral is concave, the part between the brackets in the expression (A) is positive; hence, $\frac{dp}{d\rho}$ must be positive; hence, (dp) and ($d\rho$) must both have the same sign; and, consequently, as (ρ) increases or decreases, (p) increases or decreases*.

If the spiral had been convex, the part between the brackets would have been negative; and, consequently, $\frac{dp}{d\rho}$ would have been negative; and,

---

* This immediately follows from Taylor's Theorem. Suppose (ρ) to increase by a small quantity (k), and let (p) in the mean time change its value to (p'), then

$$p' = p + \frac{dp}{d\rho} \, . \, k + \frac{d^2p}{d\rho^2} \, . \, k^2 + \&c.$$

Now, if the spiral be concave, $\frac{dp}{d\rho}$ has been shown to be positive; and, consequently, if (k) be taken so small, that the term $\frac{dp}{d\rho} \, . \, k$ is greater than the sum of all the rest, (p') must be greater than (p), that is, as (ρ) increases, (p) also must increase.

therefore, when (ρ) increased or decreased, (p) would have decreased or increased.

Cor. 3. Hence, if we know the equation between (p) and (ρ), we may easily determine, whether the spiral is concave or convex. For if, by differentiation, we find that the value of $\frac{dp}{d\rho}$ is positive, the spiral is concave, and if $\frac{dp}{d\rho}$ be negative, the spiral is convex.

Ex. 1. In the parabola, we know that

$$p = \sqrt{a\rho};$$

$$\therefore \frac{dp}{d\rho} = \frac{1}{2}\sqrt{\frac{a}{\rho}},$$

and is always positive. Hence, the parabola is always concave.

Ex. 2. In the logarithmic spiral

$$p = \rho \cdot \frac{1}{\sqrt{1 + \{\text{hyp. log. } (a)\}^2}};$$

$$\therefore \frac{dp}{d\rho} = \frac{1}{\sqrt{1 + \{\text{hyp. log. } (a)\}^2}},$$

and is always positive; hence, the logarithmic spiral is always concave.

Ex. 3. In the ellipse, we have, by conic sections,

$$p = b\sqrt{\frac{\rho}{2a - \rho}}; \quad \text{hence, } \frac{1}{p^2} = \frac{2a}{b^2\rho} - \frac{1}{b^2};$$

$$\therefore -\frac{dp}{p^3} = -\frac{a\,d\rho}{b^2\rho^2}; \quad \therefore \frac{dp}{d\rho} = \frac{ap^3}{b^2\rho^2},$$

and is always positive; consequently, the ellipse is always concave towards the pole.

Prop. LII.

At the points of contrary flexure in spirals, the perpendicular upon the tangent is a maximum or a minimum.

It has been already shown in the last Corollary to the last Proposition, that when the spiral is concave, (p) and (ρ) increase together; and that, when the spiral is convex, as (ρ) increases, (p) decreases. Now, suppose (ρ) to increase, then, as long as the spiral continues concave, (p) increases; but, if the spiral from being concave, afterwards becomes convex, then, as soon as the convexity begins, (p) will begin to decrease; consequently, (p) will be a maximum at the point where the change from concavity to convexity takes place, that is, at the point of contrary flexure. In the same manner it may be shown, that if the spiral be first convex, and then concave, (p) will be a minimum at the point of contrary flexure.

Cor. 1. Conversely, when (p) is either a maximum or a minimum, then [provided that (ρ) is neither a maximum nor a minimum], there is a point of contrary flexure.

Cor. 2. Hence, the points of contrary flexure may be thus determined. Find (p) in terms of (ρ), by means of the equation

$$\frac{1}{p^2} = \frac{1}{\rho^2} + \frac{d\rho^2}{\rho^4 d\theta^2},$$

from thence find the value of $\frac{dp}{d\rho}$; and the equation

$\frac{dp}{d\rho} = 0$, will give the points of contrary flexure, provided that at those points the value of $\frac{d\rho}{d\theta}$ is finite.

COR. 3. Though at a point of contrary flexure $\frac{dp}{d\rho} = 0$, yet the converse is not necessarily true; for $\frac{dp}{d\rho}$ may $= 0$; and yet the corresponding point not be a point of contrary flexure. For if, when $\frac{dp}{d\rho} = 0$, $\frac{d^2p}{d\rho^2} = 0$, and $\frac{d^3p}{d\rho^3}$ remains finite, (p) will not be a maximum or a minimum.

COR. 4. Hence, at a point of contrary flexure, $R_2 = 0$, or

$$\frac{2d\rho^2}{\rho d\theta^2} + \rho = \frac{d^2\rho}{d\theta^2}.$$

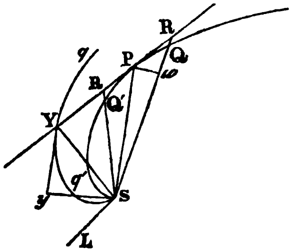
COR. 5. If at any point, the number of coefficients R_2, R_3, R_4, &c. which vanish, be odd, this point is a point of contrary flexure.

COR. 6. If at any point, the number of coefficients R_2, R_3, R_4, &c. which vanish, be even, that point is not a point of contrary flexure.

PROP. LIII.

If, in any spiral, SY *be a perpendicular upon the tangent to the point* P, qYq′ *the curve traced out by the point* Y, *and* Sy *a perpendicular upon the tangent to the curve* qYq′ *at the point* Y; *then we have this proportion*

SP : SY :: SY : Sy.



Let SP be the spiral, SL the initial line, the angle $LSP=\theta$, the angle $LSY=\theta'$, $SP=\rho$, $SY=p$, $Sy=p'$; then, since (p), (p') and (θ') are the radius vector, the perpendicular upon the tangent, and the spiral angle of the spiral qYq' respectively, we have, by (Prop. 50. Cor. 2.)

$$\frac{1}{p'^2}=\frac{1}{p^2}+\frac{dp^2}{p^4 d\theta'^2} \qquad (A).$$

Again, in the triangle SPY, we have

$$\cos PSY \text{ or } \cos(\theta-\theta')=\frac{p}{\rho},$$

$$\left(\text{and } \therefore \sin(\theta-\theta')=\sqrt{1-\frac{p^2}{\rho^2}}\right);$$

$\therefore$ differentiating, and dividing by $-\sin(\theta-\theta')$,

$$d\theta-d\theta'=\frac{p\,d\rho-\rho\,dp}{\rho^2\,.\sin(\theta-\theta')}=\frac{p\,d\rho-\rho\,dp}{\rho\sqrt{\rho^2-p^2}},$$

but $d\theta=\dfrac{p\,d\rho}{\rho\sqrt{\rho^2-p^2}}$, (by Prop. 50. Cor. 3.);

$$\therefore\ d\theta'=\frac{dp}{\sqrt{\rho^2-p^2}};$$

$$\therefore\ \frac{dp}{d\theta'}=\sqrt{\rho^2-p^2}.$$

Substituting this value in equation (A), we have

$$\frac{1}{p'^2}=\frac{1}{p^2}+\frac{\rho^2-p^2}{p^4};$$

$$\therefore \frac{p^4}{p'^2} = \rho^2; \quad \therefore \rho : p :: p : p',$$

$$\text{or } SP : SY :: SY : Sy.$$

Cor. 1. Hence, if we know the equation between the radius vector, and the perpendicular upon the tangent in the spiral SP, we may find the corresponding equation in the spiral traced out by Y.

Ex. 1. Suppose the first spiral to be the logarithmic spiral, then

$$p = \rho \cdot \frac{1}{\sqrt{1 + \{\text{hyp. log. } (a)\}^2}},$$

$$\text{but } p' : p :: p : \rho$$

$$:: 1 : \sqrt{1 + \{\text{hyp. log. } (a)\}^2};$$

$$\therefore p' = p \cdot \frac{1}{\sqrt{1 + \{\text{hyp. log. } (a)\}^2}}.$$

That is, the second spiral is also a logarithmic spiral similar to the first.

Ex. 2. Suppose the first spiral to be a parabola. Here $p = \sqrt{a\rho}$; hence, $p' = a$; and, consequently, the line traced out by Y, must be a straight line.

Ex. 3. Suppose the first spiral to be a circle, whose pole lies in the circumference. Let $(a) =$ the diameter of this circle, then

$$p = \frac{\rho^2}{a}, \quad \text{but } \rho = \frac{p^2}{p'};$$

$$\text{hence, } p = \frac{p^4}{ap'^2}; \quad \therefore p'^2 = \frac{p^3}{a}.$$

Ex. 4. Let the first spiral be the equiangular hyperbola, having its centre for its pole. Here it may easily be shown from conic sections, that $\rho = \frac{a^2}{p}$, but, by this proposition,

$$p' : p :: p : \rho \text{ or } \frac{a^2}{p};$$

$$\text{hence, } p' = \frac{p^3}{a^2}.$$

This curve is sometimes termed the Lemniscate.

From the expression $p' = \frac{p^3}{a^2}$, we may find the equation between the spiral angle, and the radius vector (p). In general,

$$\frac{1}{p'^2} = \frac{1}{p^2} + \frac{dp^2}{p^4 . d\theta^2}.$$

Hence, in this case,

$$\frac{a^4}{p^6} = \frac{1}{p^2} + \frac{dp^2}{p^4 d\theta^2};$$

$$\therefore \; d\theta = \pm \frac{p\,dp}{\sqrt{a^4 - p^4}}.$$

Where, whether we are to take the positive or the negative sign, depends upon the position we assume for the initial line. Draw a straight line bisecting the right angle, contained by the asymptotes of the hyperbola, and assume this for the initial line. It is manifest, that when (ρ) coincides with this line, (p) will also coincide with it, and both (p) and (ρ) will $= a$.

Now assume $p^2 = a^2 x$, then $p\,dp = \frac{a^2}{2} . dx$;

$$\therefore \; d\theta = -\frac{1}{2} . \frac{dx}{\sqrt{1 - x^2}},$$

$$\text{but } d.\cos^{-1}x = \frac{-dx}{\sqrt{1-x^2}};$$

$$\therefore\ d\theta = \frac{1}{2}.d\cos^{-1}x = d.\frac{\cos^{-1}x}{2}$$

$$= d.\frac{1}{2}\cos^{-1}\frac{p^2}{a^2}.$$

Hence, (θ) and $\frac{1}{2}\cos^{-1}\frac{p^2}{a^2}$ have the same differential; and, consequently, their difference is constant, now when $p=a$, $\cos^{-1}\frac{p^2}{a^2}=0$; but at this point $\theta=0$; hence, the difference between θ and $\frac{1}{2}\cos^{-1}\frac{p^2}{a^2}=0$; and, consequently, this difference always $=0$; hence, we have

$$\cos^{-1}\frac{p^2}{a^2} = 2\theta, \text{ or } \frac{p^2}{a^2} = \cos 2\theta;$$

$$\text{and } \therefore\ p^2 = a^2.\cos 2\theta,$$

which is the equation to the lemniscate.

From this we may also find the equation to the lemniscate, considered as a curve, referred to rectangular co-ordinates.

Let the straight line, which we assumed for its initial line, as a spiral, be now assumed for its horizontal axis, and let the centre of the hyperbola be the first point, from which the co-ordinates are measured.

Let the abscissa $=x$, the ordinate $=y$, then we have the following equations,

$$p^2 = x^2 + y^2,$$

$$x = p \,.\cos\theta,$$

$$y = p \,.\sin\theta,$$

$$\text{but } \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \frac{x^2}{p^2} - \frac{y^2}{p^2}.$$

$$\text{or } \frac{p^2}{a^2} = \frac{x^2}{p^2} - \frac{y^2}{p^2};$$

$$\therefore\ p^4 \text{ or } (x^2+y^2)^2 = a^2.(x^2-y^2).$$

PROP. LIV.

To find the differential coefficient of the length of a spiral.

Let *SPQ*, (See Fig. Prop. 53. p. 164.) be the spiral, *SP* its radius vector $=\rho$, *YPR* a tangent to the point *P*, *SQR* another radius vector, making with *SP* an angle $PSR = (\iota)$, join *PQ*, then the arc *PQ* is greater than the chord *PQ*; and, therefore, greater than $PR - QR$, and it is also less than $PR + QR$, but, (by Prop. 49. Cor. 2.).

$$PR = p\left\{B_1(\iota) + B_2 \,.\frac{\iota^2}{1\,.\,2} + B_3 \,.\frac{\iota^3}{1\,.\,2\,.\,3} + \&\text{c.}\right\},$$

and, (by Prop. 50. Cor. 4.),

$$QR = R_2 \,.\frac{\iota^2}{1\,.\,2} + R_3 \,.\frac{\iota^3}{1\,.\,2\,.\,3} + \&\text{c.};$$

$$\therefore\ PR + QR = p\,B_1(\iota) + (pB_2 - R_2)\frac{\iota^2}{1.2} + \&\text{c.} \quad (A),$$

$$\text{and } PR - QR = pB_1(\iota) + (pB_2 - R_2)\frac{\iota^2}{1.2} + \&\text{c.} \quad (B).$$

Now, if the arc $SP=s$, we have the arc PQ, which is its increment, while the spiral angle increases by (ι), equal to the series

$$\frac{ds}{d\theta}\cdot(\iota)+\frac{d^2s}{d\theta^2}\cdot\frac{\iota^2}{1\,.\,2}+\&\text{c.} \qquad (C).$$

But, since the arc PQ lies between $PR+RQ$, and $PR-RQ$, the series (C) lies between the series (A) and (B); hence, (by Prop. 13. Cor. 1.), we have

$$\frac{ds}{d\theta}=pB_1=p\,.\sec^2\theta'=p\,.\sec\theta'\sqrt{1+\tan^2\theta'}$$

$$=\rho\sqrt{1+\frac{d\rho^2}{\rho^2d\theta^2}}=\sqrt{\frac{d\rho^2+\rho^2d\theta^2}{d\theta^2}}.$$

COR. 1. Hence, $ds=\sqrt{d\rho^2+\rho^2d\theta^2}$.

Ex. 1. To find the length of the logarithmic spiral.

Here $\rho=a^\theta$; $\therefore\ \theta\,.\,\text{hyp. log.}\,(a)=\text{hyp. log.}\,\rho$;

$$\therefore\ d\theta=\frac{1}{\text{hyp. log. }a}\cdot\frac{d\rho}{\rho}.$$

But $ds=\sqrt{d\rho^2+\rho^2d\theta^2}=\sqrt{d\rho^2+\dfrac{d\rho^2}{\{\text{hyp. log. }(a)\}^2}}$

$$=\frac{d\rho}{\text{hyp. log. }(a)}\cdot\sqrt{1+\{\text{hyp. log. }(a)\}^2}$$

$$=d\,.\left\{\frac{\rho\cdot\sqrt{1+\{\text{hyp. log. }(a)\}^2}}{\text{hyp. log. }(a)}\right\};$$

hence, s and $\dfrac{\rho\cdot\sqrt{1+\{\text{hyp. log. }(a)\}^2}}{\text{hyp. log. }(a)}$ are two functions of (θ), which have the same differential coefficient; and, therefore, their difference is constant. Now, if we suppose that when $\rho=0$, $s=0$, we have

$$s = \rho \,.\, \frac{\sqrt{1+\{\text{hyp. log. }(a)\}^2}}{\text{hyp. log. }(a)},$$

$$\text{but } p = \frac{\rho}{\sqrt{1+\{\text{hyp. log. }(a)\}^2}};$$

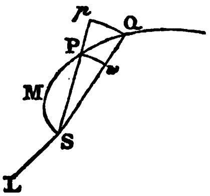
$$\text{hence, hyp. log. }(a) = \frac{\sqrt{\rho^2 - p^2}}{p};$$

$$\text{and, consequently, } s = \frac{\rho^2}{\sqrt{\rho^2 - p^2}};$$

and, therefore, *PY* : *PS* :: *PS* : the length of the curve.

Prop. LV.

To find the differential of the area of a spiral.



Let *SPQ* be a spiral, *SL* the initial line, the angle $LSP = \theta$, the angle $PSQ = \iota$, $SP = \rho$, and *SQ* another radius vector, from *S* as a centre at the distances *SP* and *SQ*, describe two circular arcs $P\pi$, *Qp*. Then the area of the sector $SP\pi = \frac{\rho^2 . \iota}{2}$, (Prop. 39. Cor. 3.). Also, the area of the sector $SQp = \frac{SQ^2 . \iota}{2}$,

$$\text{but } SQ^2 = \left(\rho + \frac{d\rho}{d\theta} . \iota + \frac{d^2\rho}{d\theta^2} . \frac{\iota^2}{1 . 2} + \&c.\right)^2$$

$$= \rho^2 + \frac{2\rho d\rho}{d\theta} . \iota + \&c.$$

Hence, the sector $SQp = \frac{\rho^2 . \iota}{2} + \frac{\rho d\rho}{d\theta} . \iota^2 + \&c.$ (*A*).

Now, the area *SMP* of the spiral, is a function of the angle *LSP* or (θ), and if we represent it by $\psi(\theta)$, we have

$$\text{area } PSQ = \psi(\theta+\iota) - \psi(\theta)$$

$$= f.\psi(\theta).\iota + f^2.\psi(\theta).\frac{\iota^2}{1.2} + \&c. \quad (B).$$

Now, since the area *PSQ* lies between the sectors $SP\pi$, SQp, the series (B) will lie between the series (A), and the quantity $\frac{\rho^2.\iota}{2}$; hence,

$$f.\psi(\theta) \text{ or } \frac{d.\psi(\theta)}{d\theta} = \frac{\rho^2}{2};$$

$$\therefore d.\psi(\theta) = \frac{\rho^2 d\theta}{2}.$$

SECT. VII.

ON THE CURVATURE OF CURVES.

◆

Def. A curve is said to have at any point a greater, or a less degree of curvature, according as it deflects more or less from the tangent to that point.

It is manifest, that the same curve may have different degrees of curvature at different points. In the hyperbola, the curvature is greatest at the vertex; as the distance of any point from the vertex increases, the form of the curve manifestly approximates to that of a straight line; and, consequently, its deviation from its tangent continually decreases, that is, its curvature becomes less and less.

In the following propositions, we shall assume these four axioms.

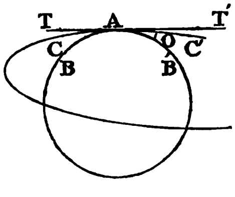
1. The curvature of the circumference of a circle is the same throughout.

The truth of this is manifest. For at every point the deviation from the tangent is the same.

2. Of two circles, the circumference of that which has the greater radius, has a less degree of curvature. The truth of this will immediately appear, if we suppose the circles to touch each other internally, for in that case, the smaller circle falls entirely within the other, and if a tangent be drawn touching them at the point, where they touch each other, the

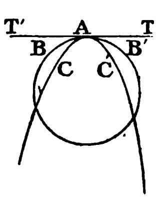
deviation of the circumference of the inner circle from this tangent, is manifestly greater than the deviation of the circumference of the other circle.

3. If a circle and a curve have at any point a common tangent, and if the circle at the touching point falls entirely within the curve, the curvature of the circle at that point cannot be less than that of the curve.



Thus, if the line *TAT'* touch the curve *CAC'*, and the circle *BAB'* at the point *A*, and the two arcs *AB*, *AB'* of the circle measured from *A*, fall both within the curve *CAC'*, it is manifest, that if the deviation of the circle *BAB* at the point *A* from the tangent *TAT'* be not greater*, it certainly cannot be less than that of the curve at the same point; that is, the curvature of the circle at *A*, is not less than that of the curve.

4. If a circle and a curve have at any point a common tangent, and if at that point the curve falls entirely within the circle, the curvature of the circle cannot be greater than that of the curve at that point.



Suppose the straight line *T'AT* to touch the circle *BAB'*, and the curve *CAC'* at the point *A*, and let that part of the curve *CAC'*, which is immediately adjacent to the point *A*, fall entirely within the circle; then it is manifest, that if at this point, the deviation of the curve from the tangent be not greater, at least it is not less than the deviation of the circle; and, consequently, the curvature of the circle cannot be greater than that of the curve.

---

* A circle may be drawn touching an ellipse at the extremity of the axis minor, which, at the point of contact, shall fall entirely within the ellipse, and yet have its curvature at that point not greater than, but equal to, the curvature of the ellipse. If, however, the curvature of a circle, at that point, be less than that of the ellipse, the circle must fall externally.

PROP. LVI.

Having given the first and second differential coefficients of the ordinate of a circle, considered as a function of the abscissa; to find the radius of the circle.

Let the first and second differential coefficients of the ordinate $= (p)$ and (q) respectively; let $r =$ the radius of the circle, and let (α) and $(\beta) =$ the co-ordinates of the centre; then we have, (by Prop. 10.)

$$(y-\alpha)^2 + (x-\beta)^2 = r^2;$$

differentiating, and dividing by dx,

$$(y-\alpha)\frac{dy}{dx} + (x-\beta) = 0;$$

differentiating again, and dividing by dx,

$$\frac{dy^2}{dx^2} + \frac{d^2y}{dx^2} \cdot (y-\alpha) + 1 = 0.$$

Eliminating the quantities $(y-\alpha)$, $(x-\beta)$, and substituting (p) and (q) for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, we have

$$r^2 = \frac{(1+p^2)^3}{q^2};$$

$$\therefore\ r = \pm\frac{(1+p^2)^{\frac{3}{2}}}{q}.$$

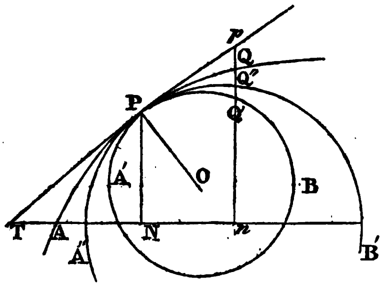
Where the negative sign is generally used.

Cor. 1. Hence, $q = \pm \frac{(1+p^2)^{\frac{3}{2}}}{r}$.

Cor. 2. Hence, there cannot be two unequal circles, the first and second differential coefficients of whose ordinates at any point are respectively equal.

Cor. 3. Hence, we may describe a circle touching a given straight line at a given point, and having the second differential coefficient of its ordinate at that point equal to a given quantity.

Let *TPp* be the given straight line, *ANn* the horizontal axis to which the circle is to be referred,



P the given point, (q') the given quantity; take $p' = \tan PTN$, from *P* draw *PO* perpendicular to *TPp*, and $= \frac{(1+p'^2)^{\frac{3}{2}}}{-q'}$; and from centre *O*, with radius *PO*, describe the circle *A'PQ'*. Then, if (p) and (q) be the first and second differential coefficients of the ordinate *PN*, we have, (by this Prop.)

$$PO = \frac{(1+p^2)^{\frac{3}{2}}}{-q}, \quad \text{but } PO = \frac{(1+p'^2)^{\frac{3}{2}}}{-q'};$$

$$\text{hence, } \frac{(1+p^2)^{\frac{3}{2}}}{-q} = \frac{(1+p'^2)^{\frac{3}{2}}}{-q'}.$$

Now $p = \tan PTN$, (by Prop. 41.) $= p'$;

hence, $q = q'$;

and, consequently, $A'PQ'$ will be the circle required.

Def. If a circle $A''PQ''$ and a curve APQ*, have a common tangent at any point P, and lie upon the same side of it, and if the curvature of the curve at P is equal to that of the circumference of the circle, the circle $A''PQ''$ is said to be the circle of curvature to the curve at P.

Def. If PO be the radius of this circle, it is said to be the radius of curvature at P.

Def. If O be the centre of this circle, it is said to be the centre of curvature to the point P.

Prop. LVII.

If, at any point, a circle and a curve have a common tangent, then, according as at that point, the second differential coefficient of the ordinate of the circle is less or greater than that of the ordinate of the curve, the circle will, at that point, lie below or above the curve.

Let APQ, $A'PQ'$ be the curve, and the circle, having the common tangent TPp; let ANn be the horizontal axis, and A the first point, from which the abscissa is measured in each. Let $AN = x$, $PN = y$; take $Nn = h$, and erect the ordinate nQp, cutting the curve and the circle in the points Q and Q'. Let p, q, r, &c. be the first, second, third, &c. differential coefficients of (y), when considered as the ordinate of the curve, and p', q', r', &c. when considered as the ordinate of the circle; then we have

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* See the figure to Prop. 57. Cor. 3.

$$Qn = y + ph + q . \frac{h^2}{1 . 2} + r . \frac{h^3}{1 . 2 . 3} + \&c.$$

and $$Q'n = y + p'h + q' . \frac{h^2}{1 . 2} + r' . \frac{h^3}{1 . 2 . 3} + \&c.$$

Now, $p = p'$, (by Prop. 41. Cor. 8.); hence,

$$Qn - Q'n = (q - q') . \frac{h^2}{1.2} + (r - r') . \frac{h^3}{1.2.3} + \&c. \quad (A).$$

Now, if (q') be less than q, and (h) sufficiently diminished, the series (A) is positive; hence, Qn is greater than $Q'n$; and, therefore, the arc PQ' lies below PQ; in the same manner it may be shown, that a small arc of the circle on the other side of P, lies below the arc of the curve. Hence, the circle falls below the curve.

In the same way it may be shown, that if (q) be greater than (q), the circle falls above the curve.

Cor. 1. If the curve be concave, and q' less than q, then, since the curve and the circle lie both beneath the tangent, and the circle lies beneath the curve, the circle must fall within the curve; and, consequently, its curvature cannot be less than that of the curve.

Cor. 2. If the curve be concave, and (q') greater than (q), the curvature of the circle cannot be greater than that of the curve.

Cor. 3. If the curve be convex, and (q') less than (q), then, since the curve and the circle lie both above the tangent, and the circle lies beneath the curve, the curve must fall within the circle; and, therefore, the curvature of the circle cannot be greater than that of the curve.

Cor. 4. If the curve be convex, and (q') greater than (q), the curvature of the circle cannot be less than that of the curve.

Prop. LVIII.

If a circle and a curve have, at any point, a common tangent, and if, at that point, the second differential coefficients of their ordinate be the same in both, the circle will be the circle of curvature to the curve.*

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* The circle of curvature to any point in a curve, has sometimes been defined to be that circle, whose contact with the curve is such, that no other circle can be drawn, so as to lie between them.

From this definition it may easily be proved, that if the circle $A'PQ'$ be the circle of curvature, (q') must equal (q). For if not, let (q') be less than (q), and (q'') some other quantity lying between them. Describe the circle $A''PQ''$, having the second differential coefficient of its ordinate at $P = q''$. Then we have

$$Qn = y + ph + q \cdot \frac{h^2}{1 \cdot 2} + r \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$Q'n = y + p'h + q' \cdot \frac{h^2}{1 \cdot 2} + r' \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

and

$$Q''n = y + p''h + q'' \cdot \frac{h^2}{1 \cdot 2} + r'' \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

But $p = p' = p''$, because the three curves have a common tangent. Hence,

$$Qn - Q''n = (q - q'') \cdot \frac{h^2}{1 \cdot 2} + (r - r'') \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \&c. \qquad (M),$$

$$Q''n - Q'n = (q'' - q') \cdot \frac{h^2}{1 \cdot 2} + (r'' - r') \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \&c. \qquad (N).$$

Now, ($q - q''$) and ($q'' - q'$) are both positive; hence, if (h) be sufficiently diminished, $Qn - Q''n$ and $Q''n - Q'n$ must be both positive; hence, $Q''n$ will be less than Qn, and greater than $Q'n$, or Q'' will lie between Q and Q'; that is, the circle $A''PQ''$ lies between the circle $A'PQ'$ and the curve APQ; which is contrary to the supposition; hence, q' must $= q$.

Suppose the curve to be concave, then, using the same figure, construction, and notation as before, let, if possible, the circle $A'PQ'$ be the circle of curvature to the curve at P, (q') being less than (q). Take (q'') a quantity lying between (q) and (q'), and, (by Cor. 3. Prop. 56.), describe another circle $A''PQ''$, touching the line TPp at P, and having the second differential coefficient of its ordinate $= q''$. Then, (by Prop. 57. Cor. 1.) since (q'') is less than q, the curvature of the circle $A''PQ''$ is not less than that of the curve at P.

Again, since (q') is less than (q''), the circle $A'PQ'$ falls beneath, and, consequently, within the circle $A''PQ''$, and, therefore, (Axiom 2.) the curvature of $A'PQ'$ is greater than that of $A''PQ''$, but the curvature of $A''P.Q''$ is not less than that of the curve; therefore the curvature of $A'PQ'$ is greater than that of the curve at P.

Hence it appears, that every circle $A'PQ'$, in which (q') is less than (q), has a greater degree of curvature than the curve at P; and, in the same manner it may be shown, that if in the circle $A'PQ'$, (q') had been greater than (q), this circle would have had a less degree of curvature, than the curve at P; consequently, $A'PQ'$ cannot be the circle of curvature, unless $q' = q$. In the same manner this proposition may be proved, when the curve is convex.

Cor. 1. Hence, if R be the radius of curvature, we have

$$R = \frac{(1+p^2)^{\frac{3}{2}}}{-q}.$$

For PO, the radius of the circle $A'PQ' = \dfrac{(1+p'^2)^{\frac{3}{2}}}{-q'}$, and if this circle is the circle of curvature, $PO=R$, $p=p'$, and $q=q'$;

$$\text{hence, } R=\frac{(1+p^2)^{\frac{3}{2}}}{-q}.$$

Ex. 1. In a parabola, $y=2a^{\frac{1}{2}}x^{\frac{1}{2}}$;

$$\therefore\ p \text{ or } \frac{dy}{dx}=\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}},$$

$$\text{and } q \text{ or } \frac{d^2y}{dx^2}=-\frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}.$$

$$\text{Hence, } R=\left(1+\frac{a}{x}\right)^{\frac{3}{2}}\div\frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}=\frac{2.(x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

Ex. 2. In the logarithmic curve, $y=a^x$;

hence, $p=A.a^x$, and $q=A^2.a^x$;

$$\text{hence, } R=\frac{(1+A^2a^{2x})^{\frac{3}{2}}}{-A^2.a^2}=\frac{(1+A^2y^2)^{\frac{3}{2}}}{-A^2y}.$$

Ex. 3. In the cycloid, we have

$$y=\sqrt{2ax-x^2}+a.\text{ver sin}^{-1}.\frac{x}{a};$$

$$\therefore\ p=\frac{a-x}{\sqrt{2ax-x^2}}+\frac{a}{\sqrt{2ax-x^2}}$$

$$=\frac{2a-x}{\sqrt{2ax-x^2}}=\frac{1}{x}.\sqrt{2ax-x^2};$$

$$\therefore\ 1+p^2=\frac{2a}{x},\quad \text{also } q=\frac{-a}{x\sqrt{2ax-x^2}}.$$

$$\text{Hence, } R=\left(\frac{2a}{x}\right)^{\frac{3}{2}}\div\frac{a}{x\sqrt{2ax-x^2}}$$

$$=2\sqrt{4a^2-2ax}.$$

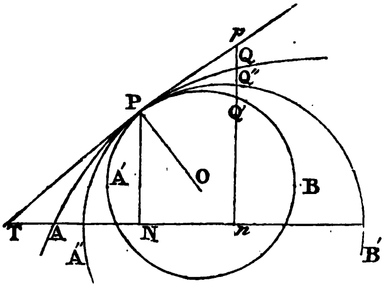
Cor. 2. No circle can be drawn at the point *P*, whose arcs adjacent to *P*, fall between the curve and the circle of curvature. Let *A'PQ'* be the circle of curvature, and *A''PQ''* any other circle. Then $q' = q$, and if (q'') be less than (q) or (q'), *A''PQ''* falls beneath both *APQ* and *A'PQ'*, (Prop. 57.); if (q'') be greater than (q) or (q'), *A''PQ''* lies above both *APQ* and *A'PQ'*; and if $q'' = q'$, *A''PQ'* coincides with *A'PQ'*, (because there cannot be two unequal circles, the first and second differentials of whose ordinates are at any point respectively equal). Hence, in none of these three cases, can the circle *A''PQ''* lie between *A'PQ'* and the curve *APQ*.

Cor. 3. At a point of contrary flexure, the radius of curvature either $= 0$, or becomes infinite. For at these points (q) either $= 0$, or is infinite.

Prop. LIX.

If the circle of curvature to any point of a curve be drawn, it cuts the curve.

Using the same figure and construction as before, let *A''PQ''* be the circle of curvature at *P*. Assume



p, *q*, *r*, *s*, &c. equal to the differential coefficients of *PN*, when considered as an ordinate of the curve, and p', q', r', s', &c., when considered as an ordinate of the circle. Then we have

$$Qn = y + ph + q \cdot \frac{h^2}{1.2} + r \cdot \frac{h^3}{1.2.3} + \text{\&c.}$$

$$Q''n = y + p'h + q' \cdot \frac{h^2}{1.2} + r' \cdot \frac{h^3}{1.2.3} + \text{\&c.},$$

and, therefore, since $p = p'$, and $q = q'$,

$$Qn - Q''n = (r - r') \cdot \frac{h^3}{1.2.3} + (s - s') \cdot \frac{h^4}{1.2.3.4} + \text{\&c.}$$

First, let (r) be greater than (r'), then, if (h) be sufficiently small, Qn must be greater than $Q''n$; and, therefore, Q will lie above Q''; and, therefore, the arc PQ above the arc PQ''.

Now, if Nn had been measured on the other side of N, and Qn, $Q''n$ then been the ordinates of the curve and circle to the abscissa An, we should have had

$$Qn = y - ph + q \cdot \frac{h^2}{1.2} - r \cdot \frac{h^3}{1.2.3} + \text{\&c.}$$

$$Q''n = y - p'h + q' \cdot \frac{h^2}{1.2} - r' \cdot \frac{h^3}{1.2.3} + \text{\&c.};$$

and, therefore,

$$Qn - Q''n = (r' - r) \cdot \frac{h^3}{1.2.3} + (s - s') \cdot \frac{h^4}{1.2.3.4} + \text{\&c.}$$

But, since $r' - r$ is negative, $Qn - Q''n$ must be negative, when (h) is small; hence, Qn will be less than $Q''n$; and, therefore, Q will lie below Q''; and, consequently, the arc PQ below the arc PQ''.

Since, then, an arc of the curve, when measured on one side of the ordinate PN, falls above the circle

$A''PQ''$, and when measured on the other, falls below, the circle and the curve must intersect each other at P.

In the same manner the proposition may be proved, if (r) be less than (r').

COR. 1. This proposition will not hold good, if $r = r'$, for in that case, when Nn is measured to the right of N, we have

$$Qn - Q''n = (s - s') \cdot \frac{h^4}{1.2.3.4} + (t - t') \cdot \frac{h^5}{1.2.3.4.5} + \&\text{c.},$$

and when Nn is measured to the left of N, we have

$$Qn - Q''n = (s - s') \cdot \frac{h^4}{1.2.3.4} - (t - t') \cdot \frac{h^5}{1.2.3.4.5} + \&\text{c.}$$

Hence, when (h) is sufficiently diminished, $Qn - Q''n$ is, in both cases, either positive or negative, according as (s) is greater or less than (s'); and, consequently, the curve APQ lies either above or below the circle $A''PQ''$, on both sides of P. At this point the curvature of the curve is either a maximum or a minimum. Let $R =$ the radius of curvature; then R may be considered as a function of (x), and we have

$$R = \frac{(1 + p^2)^{\frac{3}{2}}}{-q};$$

$$\therefore \frac{dR}{dx} = -\frac{3 \cdot (1 + p^2)^{\frac{1}{2}} \cdot p\,dp}{q\,dx} + \frac{dq \cdot (1 + p^2)^{\frac{3}{2}}}{q^2\,dx}$$

$$= (1 + p^2)^{\frac{1}{2}} \cdot \left\{ -3p + \frac{r \cdot (1 + p^2)}{q^2} \right\} \qquad (A),$$

because $q = \dfrac{dp}{dx}$, and $r = \dfrac{dq}{dx}$.

Again, in the circle of curvature, we have

$$(x-\alpha)^2+(y-\beta)^2=R^2;$$

$$\therefore\ (x-\alpha)+p.(y-\beta)=0;$$

$$\therefore\ 1+p^2+q.(y-\beta)=0,$$

and, by another differentiation,

$$2pq+qp+r.(y-\beta)=0,$$

eliminating $(y-\beta)$ from these two last equations, we have

$$3pq=\frac{r.(1+p^2)}{q},$$

combining this equation with the equation (*A*), we have

$$\frac{dR}{dx}=0;$$

and, consequently, (unless $\frac{d^2R}{dx^2}$ be also equal to nothing, which will be the case when $s=s'$), *R* is either a maximum or a minimum.

COR. 2. This proposition will again hold good, if, when $r=r'$, s also $=s'$. For in that case, when *Nn* is measured to the right of *N*, we have

$$Qn-Q''n=(t-t').\frac{h^5}{1.2\ldots\ldots5}$$

$$+(u-u').\frac{h^6}{1.2\ldots\ldots6}+\&c.$$

and when *Nn* is measured to the left of *N*,

$$Qn - Q''n = -(t-t') \cdot \frac{h^5}{1 \cdot 2 \ldots\ldots 5}$$

$$+ (u-u') \cdot \frac{h^6}{1 \cdot 2 \ldots\ldots 6} + \&c.$$

Here, when (h) is sufficiently diminished, we may prove, as before, that the curve on one side of P lies within, and on the other without the circle $A''PQ''$. In this case R is neither a maximum nor a minimum*.

---

* It may easily be shown, by a general demonstration, that when the circle of curvature does not cut the curve, the curvature of the curve at that point is either a maximum or a minimum.

Let p, q, r, s, t, &c. be the first, second, third, fourth, fifth, &c. differential coefficients of the ordinate of the curve. Then, since R the radius of curvature $= \frac{(1+p^2)^{\frac{3}{2}}}{-q}$, it will appear, by differentiation, that $\frac{dR}{dx}$ involves (p), (q), (r), and is, consequently, a function of these three quantities, and may, therefore, be represented by $\phi \cdot (p, q, r)$.

Similarly, it may be shown, that

$$\frac{d^2R}{dx^2} = \phi'(p, q, r, s),$$

$$\text{and} \quad \frac{d^3R}{dx^3} = \phi''(p, q, r, s, t),$$

and so on for any number of differential coefficients. Again, let p', q', r', s', t', &c. be the differential coefficients of the ordinate of the circle of curvature; then we have

$$R = \frac{(1+p'^2)^{\frac{3}{2}}}{-q'},$$

and, by successive differentiations,

PROP. LX.

Having given the equation to one curve, to find the equation to a curve, which is the locus of all the centres of curvature to the different points of the given curve.

Let P be a point in the given curve AP, PN and AN the ordinate and abscissa to that point $=(y)$

$$0=\phi\,.\,(p', q', r'),$$

$$0=\phi'(p', q', r', s'),$$

$$0=\phi''(p', q', r', s', t').$$

Now, $p'=p$, and $q'=q$; and, consequently, if $r=r'$, we have

$$\phi\,.\,(p, q, r)=\phi\,.\,(p', q', r');$$

$$\therefore\ \frac{dR}{dx}=0.$$

Hence, in this case, where the circle does not cut the curve, R is either a maximum or a minimum.

If, however, when $r=r'$, s also $=s'$, we have

$$\phi\,.\,(p, q, r)=\phi\,(p', q', r'),$$

$$\text{and}\quad \phi'(p, q, r, s)=\phi'(p', q', r', s');$$

$$\therefore\ \frac{dR}{dx}=0, \quad \text{and}\ \frac{d^2R}{dx^2}=0.$$

Hence, in this case, where the circle does cut the curve, R is neither a maximum nor a minimum.

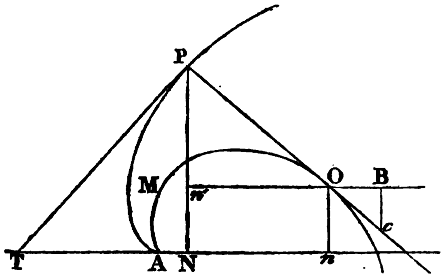
If, however, $t=t'$, we have again

$$\frac{dR}{dx}=0, \quad \frac{d^2R}{dx^2}=0, \quad \text{and}\ \frac{d^3R}{dx^3}=0;$$

and, consequently, in this case, where the circle does not cut the curve, R is either a maximum, or a minimum.

In the same way we may proceed, whatever be the number of differential coefficients in the curve, which become equal to the corresponding ones in the circle of curvature.

and (x) respectively; let (p), (q) and (r) be the first, second, and third differential coefficients of (y) considered as a function of (x). Let PO the radius of curvature $= R$, On, An, the co-ordinates to the point $O = (y')$ and (x') respectively, and through O, draw $n'OB$ parallel to ANn; then we have the following equations:



$$R = \frac{(1+p^2)^{\frac{3}{2}}}{-q}, \quad \text{(by Prop. 58.)},$$

$$\text{and } On' = p \,.\, Pn', \quad \text{or } x' - x = (y - y')\, p;$$

$$\Big[\text{because } p = \frac{dy}{dx} = \tan PTA, \text{ (by Prop. 41.)}$$

$$= \tan OPN,\Big]$$

$$\text{and } R^2 = Pn'^2 + On'^2 = (y - y')^2 + (x' - x)^2.$$

From these equations we may deduce

$$y' = y + \frac{(1+p^2)}{q} \qquad (C),$$

$$x' = x - \frac{(1+p^2)p}{q} \qquad (D).$$

Here, then, we have three equations, viz., the equation to the curve AP, and the equations (C) and

(D), and if from them we can eliminate the two unknown quantities (x) and (y), we may find the equation between (x') and (y'), which will be the equation to the curve, traced out by O.

Ex. 1. To find the nature of the curve traced out by O, when AP is a parabola.

Here $y^2=4ax$; $\therefore$ $2yp=4a$, or $yp=2a$;

$$\therefore\ p^2+qy=0;\ \text{hence,}\ \frac{1+p^2}{q}=-\frac{y^3+4a^2y}{4a^2};$$

$$\text{and}\ \frac{(1+p^2)p}{q}=-\frac{y^2+4a^2}{2a}.$$

$$\text{Hence,}\ y'=y-\frac{y^3+4a^2y}{4a^2}=-\frac{y^3}{4a^2};$$

$$\therefore\ y=-\sqrt[3]{4}\,.\,a^{\frac{2}{3}}y'^{\frac{1}{3}},$$

$$\text{and}\ x'=x+\frac{y^2+4a^2}{2a}$$

$$=x+2a+\frac{y^2}{2a}=3x+2a;$$

$$\text{and, therefore,}\ x=\frac{x'-2a}{3}.$$

Substituting these values of (y) and (x) in the equation $y^2=4ax$, we have

$$\sqrt[3]{16}\,.\,a^{\frac{4}{3}}y'^{\frac{2}{3}}=\frac{4a}{3}\,.\,(x'-2a),$$

and cubing this equation,

$$16\,.\,a^4y'^2=\frac{64a^3}{27}\,.\,(x'-2a)^3;$$

$$\therefore\ y'^2 = \frac{4}{27a}\cdot(x' - 2a)^3$$

$$= \frac{x''^3}{b},$$

where x'' is assumed $= x' - 2a$, and $b = \frac{27a}{4}$. Hence, the curve *MO* is the semi-cubical parabola.

Cor. 1. The radius of curvature *PO* is the tangent to the curve *MO*. (See Fig. Prop. 60.).

For since $y' = y + \frac{(1+p^2)}{q}$,

$$\frac{dy'}{dx} = 3p - \frac{r.(1+p^2)}{q^2};$$

because $p = \frac{dy}{dx}$, $q = \frac{dp}{dx}$, and $r = \frac{dq}{dx}$,

and since $x' = x - \frac{(1+p^2)p}{q}$,

$$\frac{dx'}{dx} = -p.\left\{3p - \frac{r.(1+p^2)}{q^2}\right\};$$

hence, $\frac{dy'}{dx'} = -\frac{1}{p}$.

Now, $\tan POB = -\tan POn' = -\tan TPN$

$$= -\cot PTN = -\frac{1}{\tan PTN}$$

$$= -\frac{1}{p} = \frac{dy'}{dx'}.$$

Hence, *PO* is a tangent to the curve *MO* at *O*.

Cor. 2. If the curve *AP*, and the curve *MO* intersect each other in A, the length of the curve AMO is equal to the radius of curvature PO.

Let the length of the arc $AMO = s'$, then, (by Prop. 45. Cor. 1.), we have

$$\frac{ds'^2}{dx^2} = \frac{dy'^2}{dx^2} + \frac{dx'^2}{dx^2};$$

but, in the former corollary, we found that

$$\frac{dy'}{dx} = 3p - \frac{r.(1+p^2)}{q},$$

$$\text{and } \frac{dx'}{dx} = -p.\left\{3p - \frac{r.(1+p^2)}{q}\right\};$$

$$\therefore \frac{ds'}{dx} = -\sqrt{1+p^2}.\left\{3p - \frac{r.(1+p^2)}{q}\right\}$$

$$= \frac{dR}{dx}, \text{ (by Prop. 49. Cor. 1.)};$$

hence, R and (s') are two functions of (x), which have the same differential coefficient; and, therefore, the difference between them is constant; but, when P coincides with A, O also coincides with A; and, therefore, their difference then $= 0$; consequently, their difference always $= 0$; and, therefore, we have R always $= s'$, or the arc $AMO = PO$, the radius of curvature.

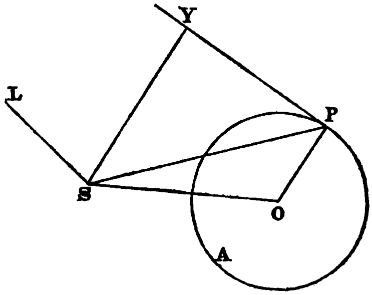
Cor. 3. Hence, it is manifest, that if a thread be wound upon the curve OMA, then, by unwinding it, and keeping it stretched, the end P may be made to trace out the curve AP. In consequence of this property, the curve AMO is said to be the evolute of the curve AP, and the curve AP the involute of AMO.

SECT. VIII.

CURVATURE OF SPIRALS.

PROP. LXI.

There cannot be two unequal circles, which, when considered as spirals, have, at any point, their radii vectores, and the first and second differential coefficients of their radii vectores, respectively equal.



LET AP be a circle whose centre is O, considered as a spiral, referred to the point S as its pole, and SL as its initial line; let the radius vector $SP=\rho$, the angle $LSP=\theta$, $SO=c$, $OP=r$, and SY the perpendicular upon the tangent at $P=p$; then we have

$$SO^2 = SP^2 + PO^2 - 2SP . PO . \cos SPO,$$

$$\text{or } c^2 = \rho^2 + r^2 - 2r\rho . \cos PSY$$

$$= \rho^2 + r^2 - 2r . p.$$

Hence, differentiating, we have

$$2\rho d\rho - 2r dp = 0, \quad \text{or } r = \frac{\rho d\rho}{dp}.$$

Similarly, if there be any other circle, considered as a spiral, and referred to the same pole and initial line, whose radius vector, radius, and perpendicular upon the tangent, are ρ', r' and p' respectively, it may be shown, that

$$r' = \frac{\rho' d\rho'}{dp'}.$$

Now, in general,

$$\frac{1}{p^2} = \frac{1}{\rho^2} + \frac{d\rho^2}{\rho^4 d\theta^2};$$

$$\therefore \frac{d\rho^2}{d\theta^2} = \frac{\rho^4}{p^2} - \rho^2 \qquad (A).$$

Hence, by differentiation,

$$\frac{d^2\rho}{d\theta^2} = \frac{2\rho^3}{p^2} - \frac{\rho^4 dp}{p^3 d\rho} - \rho \qquad (B).$$

$$\text{Similarly,} \quad \frac{d\rho'^2}{d\theta^2} = \frac{\rho'^4}{p'^2} - \rho'^2 \qquad (C),$$

$$\text{and} \quad \frac{d^2\rho'}{d\theta^2} = \frac{2\rho'^3}{p'^2} - \frac{\rho'^4 dp'}{p'^3 d\rho'} - \rho' \qquad (D).$$

Now, since by the supposition,

$$\rho = \rho', \; \frac{d\rho}{d\theta} = \frac{d\rho'}{d\theta}, \text{ and } \frac{d^2\rho}{d\theta^2} = \frac{d^2\rho'}{d\theta^2},$$

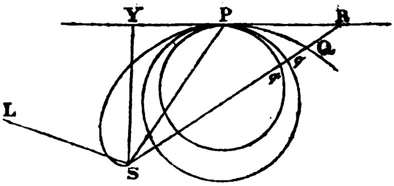
we have, from the equations (A) and (C), $p = p'$; and, consequently, from the equations (B) and (D),

$$\frac{dp}{d\rho} = \frac{dp'}{d\rho'}; \text{ hence, } \frac{\rho d\rho}{dp} = \frac{\rho' d\rho'}{dp'};$$

that is, the radii of the two circles are equal; and, consequently, the circles themselves cannot be unequal.

PROP. LXII.

If a curve SPQ *and a circle* Pq′, *considered as spirals, referred to the same pole* S, *and the same initial line* SL, *have at any point* P, *a common tangent* YPR; *then according as at that point, the second differential coefficient of the radius vector of the circle, is less or greater than that of the radius vector of the curve, the circle will lie nearer to, or further from, the pole, than the curve.*



Let $\angle LSP = \theta$, and let (ρ) and (ρ'') be the radii vectores of the curve and circle; draw another radius vector SqR cutting the circle in (q'), the curve in Q, and the tangent in R, and let $\angle PSR = \iota$. Then we have

$$SQ = \rho + \frac{d\rho}{d\theta} \cdot \iota + \frac{d^2\rho}{d\theta^2} \cdot \frac{\iota^2}{1 \cdot 2} + \&c.,$$

$$Sq' = \rho'' + \frac{d\rho''}{d\theta} \cdot \iota + \frac{d^2\rho''}{d\theta^2} \cdot \frac{\iota^2}{1 \cdot 2} + \&c.$$

Now, at the point P,

$$\rho = \rho'', \text{ and } \frac{d\rho}{d\theta} = \frac{d\rho''}{d\theta},$$

(by Prop. 50. Cor. 10.); hence, $SQ - Sq'$ is equal to the series,

$$\left\{\frac{d^2\rho}{d\theta^2} - \frac{d^2\rho''}{d\theta^2}\right\} \cdot \frac{\iota^2}{1 \cdot 2} + \left\{\frac{d^3\rho}{d\theta^3} - \frac{d^3\rho''}{d\theta^3}\right\} \cdot \frac{\iota^3}{1.2.3} + \&c.$$

Now, let $\frac{d^2\rho}{d\theta^2}$ be greater than $\frac{d^2\rho''}{d\theta^2}$, then, if ($\iota$) be sufficiently small, this series is positive; and, therefore, SQ is greater than Sq'; consequently, (q') is nearer to S than Q; and, therefore, the circular arc Pq' is nearer to the pole than PQ, the arc of the curve. In the same manner it may be shown, that a small arc of the circle Pq', measured on the other side of P, lies nearer to the pole, than the corresponding curvilinear arc; hence, when $\frac{d^2\rho''}{d\theta^2}$ is less than $\frac{d^2\rho}{d\theta^2}$, the circle lies nearer to the pole than the curve.

Cor. 1. If the spiral be concave, it falls between the tangent and the pole; and if $\frac{d^2\rho''}{d\theta^2}$ be less than $\frac{d^2\rho}{d\theta^2}$, the circle falls between the spiral and the pole; and, consequently, the circle falls within the spiral; and, therefore, its curvature is not less than that of the curve.

Cor. 2. If $\frac{d^2\rho''}{d\theta^2}$ be greater than $\frac{d^2\rho}{d\theta^2}$, the curvature of the circle is not greater than that of the curve.

Cor. 3. If the spiral be convex, and $\frac{d^2\rho''}{d\theta^2}$ less than $\frac{d^2\rho}{d\theta^2}$, the circle will fall without the spiral, (that is, between the spiral and the tangent); and, therefore, its curvature will not be greater than that of the curve.

COR. 4. If $\frac{d^2\rho''}{d\theta^2}$ be greater than $\frac{d^2\rho}{d\theta^2}$, the curvature of the circle cannot be less than that of the curve.

PROP. LXIII.

If the circle Pq *and the spiral* SPQ, *(both being considered as spirals,) have, at the point* P, *the same tangent* YPR, *and the same radius vector* SP, *and if, at that point the second differential coefficient of the radius vector be the same in both, the circle* Pq *is the circle of curvature**.

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* This proposition may easily be proved from the definition given in note, page 178.

Using the same figure and notation as before, let Pq' be the circle of curvature, and, if possible, let $\frac{d^2\rho''}{d\theta^2}$ be less than $\frac{d^2\rho}{d\theta^2}$, and Pq another circle, in which $\frac{d^2\rho'}{d\theta^2}$ lies between $\frac{d^2\rho''}{d\theta^2}$ and $\frac{d^2\rho}{d\theta^2}$, then we have

$$SQ=\rho+\frac{d\rho}{d\theta}\cdot\iota+\frac{d^2\rho}{d\theta^2}\cdot\frac{\iota^2}{1.2}+\frac{d^3\rho}{d\theta^3}\cdot\frac{\iota^3}{1.2.3}+\&c. \quad (A),$$

$$Sq=\rho'+\frac{d\rho'}{d\theta}\cdot\iota+\frac{d^2\rho'}{d\theta^2}\cdot\frac{\iota^2}{1.2}+\frac{d^3\rho'}{d\theta^3}\cdot\frac{\iota^3}{1.2.3}+\&c. \quad (B),$$

$$Sq'=\rho''+\frac{d\rho''}{d\theta^2}\cdot\iota+\frac{d^2\rho''}{d\theta^2}\cdot\frac{\iota^2}{1.2}+\frac{d^3\rho''}{d\theta^3}\cdot\frac{\iota^3}{1.2.3}+\&c. \quad (D).$$

And since, at the point P, $\rho=\rho'=\rho''$,

$$\text{and } \frac{d\rho}{d\theta}=\frac{d\rho'}{d\theta}=\frac{d\rho''}{d\theta},$$

we have

$$SQ-Sq=\left\{\frac{d^2\rho}{d\theta^2}-\frac{d^2\rho'}{d\theta^2}\right\}\cdot\frac{\iota^2}{1.2}+\left\{\frac{d^3\rho}{d\theta^3}-\frac{d^3\rho'}{d\theta^3}\right\}\cdot\frac{\iota^3}{1.2.3}+\&c. \quad (E),$$

$$Sq-Sq'=\left\{\frac{d^2\rho'}{d\theta^2}-\frac{d^2\rho''}{d\theta^2}\right\}\cdot\frac{\iota^2}{1.2}+\left\{\frac{d^3\rho}{d\theta^3}-\frac{d^3\rho'}{d\theta^3}\right\}\cdot\frac{\iota^3}{1.2.3}+\&c. \quad (F).$$

Now,

Describe the circle Pq', touching the tangent YPR in the point P, and falling entirely within the circle Pq; let the radii vectores of the spiral, and the two circles Pq, Pq' be ρ, ρ', ρ'', respectively; the perpendiculars upon the tangents, p, p', p'', and let the $\angle LSP$ which $= \theta$, be the spiral angle to them all. Then, since the circle Pq' falls within the circle Pq, $\frac{d^2\rho'}{d\theta^2}$ cannot be less than $\frac{d^2\rho''}{d\theta^2}$, (Prop. 62.

Now, if (ι) be small, since $\frac{d^2\rho'}{d\theta^2}$ is less than $\frac{d^2\rho}{d\theta^2}$, and greater than $\frac{d^2\rho''}{d\theta^2}$, both these series are positive; hence, Sq is less than SQ, and greater than Sq'; and, therefore, the point (q) lies between Q and (q'); and, consequently, the arc Pq lies between the arc PQ, and the circular arc Pq'. The same may be proved, if the arcs PQ, Pq and Pq', had been measured on the other side of P. Hence, the circle Pq lies between the spiral and the circle Pq'; and, consequently, Pq' cannot be the circle of curvature.

In the same manner it may be shown, that Pq' cannot be the circle of curvature, when $\frac{d^2\rho''}{d\theta^2}$ is greater than $\frac{d^2\rho}{d\theta^2}$. Hence, if Pq' be the circle of curvature $\frac{d^2\rho''}{d\theta^2}$ must equal $\frac{d^2\rho}{d\theta^2}$.

Conversely, if $\frac{d^2\rho''}{d\theta^2} = \frac{d^2\rho}{d\theta^2}$, Pq' must be the circle of curvature. Draw any other circle Pq touching YPR at P. Then if $\frac{d^2\rho'}{d\theta^2}$ be less than $\frac{d^2\rho}{d\theta^2}$ or $\frac{d^2\rho''}{d\theta^2}$, the series ($E$) is positive; and the series (F) negative; and, therefore, Sq is less than either SQ or Sq'; and, consequently, Pq does not lie between PQ and Pq'. In the same manner it may be shown, that Pq will not lie between PQ and Pq', when $\frac{d^2\rho'}{d\theta^2}$ is greater than $\frac{d^2\rho}{d\theta^2}$. And if $\frac{d^2\rho'}{d\theta^2} = \frac{d^2\rho}{d\theta^2}$, the circles Pq, Pq' must coincide, (Prop. 61.) Hence, in neither of these three cases can the circle Pq lie between the circle Pq' and the spiral arc PQ. Hence, Pq' must be the circle of curvature.

Cor. 1.), and it cannot be equal to $\frac{d^2\rho''}{d\theta^2}$, (Prop. 61.); hence, $\frac{d^2\rho'}{d\theta^2}$ is greater than $\frac{d^2\rho''}{d\theta^2}$, but by the supposition, $\frac{d^2\rho}{d\theta^2}=\frac{d^2\rho'}{d\theta^2}$; hence, $\frac{d^2\rho}{d\theta^2}$ is greater than $\frac{d^2\rho''}{d\theta^2}$, and, consequently, (Prop. 62. Cor. 1.) the curvature of the circle *Pq′* is not less than that of the spiral at the point *P*.

Now, if another circle be drawn touching *YPR* in *P*, so as to fall within the circle *Pq*, and without the circle *Pq′*, it may be shown, as before, that the curvature of this circle, (which is less than the curvature of *Pq′*), is not less than that of the spiral at *P*. Hence, the curvature of *Pq′* is greater than that of the spiral at *P*.

Thus it appears, that every circle *Pq′*, which has a greater curvature than *Pq*, has also a greater curvature than that of the spiral at *P*; and in the same way it may be shown, that every circle, which has a less curvature than *Pq*, and, consequently, lies without *A*, has also a less curvature than that of the spiral at *P*. Hence, no circle, but *Pq*, can have the same degree of curvature, that the spiral has at *P*; and, consequently, *Pq* is the circle of curvature.

Cor. 1. Hence, the radius of curvature to the spiral may be found. Let (r) be the radius of curvature, then it equals the radius of the circle *Pq* $=\frac{\rho' d\rho'}{dp'}$; but since at the point *P*,

$$\rho=\rho',\ \frac{d\rho}{d\theta}=\frac{d\rho'}{d\theta},\ \text{and}\ \frac{d^2\rho}{d\theta^2}=\frac{d^2\rho'}{d\theta^2},$$

we may prove, as before, in (Prop. 61.), that

$$\frac{\rho d\rho}{dp} = \frac{\rho' d\rho'}{dp'};$$

and, consequently, the radius of curvature $= \frac{\rho d\rho}{dp}$.

Cor. 2. Hence, that chord of the circle of curvature to the point P, which is drawn from P, and (produced, if necessary,) passes through the pole $S = \frac{2p d\rho}{dp}$.

Ex. 1. In the logarithmic spiral,

$$p = \frac{\rho}{\sqrt{1+A^2}}, \text{ [where } A = \text{hyp. log. } (a)\text{]};$$

$$\text{hence, } \frac{2p d\rho}{dp} = 2\rho.$$

Ex. 2. In the parabola,

$$p^2 = a\rho;$$

$$\therefore 2 \,.\, \text{hyp. log. } (p) = \text{hyp. log. } (a) + \text{hyp. log. } (\rho);$$

$$\therefore \frac{2dp}{p} = \frac{d\rho}{\rho}; \quad \therefore \frac{2p d\rho}{dp} = 4\rho.$$

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